ETH Zürich, D-INFK
HS 2017, 21. September 2017

Prof. Ueli Maurer
Marta Mularczyk

Diskrete Mathematik

Solution 1

1.1 Hilbert's Hotel

Generally, there are many valid ways to accommodate the guests. Here we only give one of the many possible solutions.

a) Hilbert asks each guest in room n to move to the room with the number $n+1$. This way, the room number 1 becomes free and can accommodate Roger Federer.

b) This time, Hilbert asks each guest in room n to move to the room with number $2n$. This way, all rooms with odd numbers become free. A newly arrived guest with number m can now stay in the room number $2m-1$.

c) Once again, Hilbert asks each guest in room n to move to the room number $2n$. Then, a guest coming in the bus i, with the number m in that bus, takes the room p_{i+1}^m, where p_i denotes the i-th prime (that is, $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$). [1]

Below we argue, a bit more formally, why the above method is correct. The argumentation implicitly uses some proof techniques, such as proof by contradiction and case distinction, as well as some facts from number theory, that will all be explained later during the lecture.

To see why the solution works, we must make sure that the following two statements are true: (1) no new guest is assigned to a room now occupied by an old guest and (2) no two new guests are assigned to the same room. The first statement follows from the fact that 2 is the only even prime (thus, any power of a prime number greater than 2 cannot be even). To justify the second statement, assume that there are two guests: one having the number m in the bus i and the other having the number n in the bus j (where $p_i \neq p_j$ or $m \neq n$), who were assigned the same room. That is, assume we have $p_i^m = p_j^n$. If it is the case that $p_i = p_j$, then it must hold that $m \neq n$ and, thus, $p_i^m \neq p_j^n$, which is a contradiction. If $p_i \neq p_j$, then we know that a prime power is divisible only by powers of the same prime which is again in contradiction with $p_i^m = p_j^n$.

Note that if Hilbert uses the method described above, some rooms are left unoccupied. There exists a more "dense" solution, where all rooms are used. Chapter 3.6 will introduce some techniques that can be used to design such a solution.

[1] A prime is a natural number, which has only two divisors: 1 and itself. Later in the lecture, we will see that there is an infinite number of primes.

1.2 Alice in the Forest of Forgetfulness

If the first statement is true, then the first brother is indeed Tweedledum, hence the second is Tweedledee and the second statement is also true. If the first statement is false, then the first brother is in fact Tweedledee and the second one is Tweedledum, hence the second statement is also false. Therefore, either both statements are true, or they are both false. There is no day of the week, on which both brothers lie. The only day, on which they both tell the truth is Sunday. Thus, both statements are true, the first brother is Tweedledum and the day of the week is Sunday.

1.3 The island

Let k be the number of collected coconuts. The conditions given in the exercise can be expressed by the following system of linear congruences: [2]

$$\begin{aligned} k &\equiv_8 1 \\ k &\equiv_7 2 \\ k &\equiv_5 3 \end{aligned}$$

One can check that the only number not greater than 250 which fulfills all of the above conditions is $k = 233$. One valid way to come up with this solution is to check all the possibilities for k, from 0 to 250. Later in the lecture, we will see a theorem (called the Chinese Reminder Theorem), which guarantees that in this case there can only be one solution smaller than $8 \cdot 7 \cdot 5 = 280$. Moreover, the proof of that theorem provides a way to compute this solution, which is faster than simply trying out all the possibilities.

Since the total number of collected coconuts is $k = 233$, each of the 5 surviving pirates got an equal amount of 46 coconuts. The monkey got the 3 remaining coconuts.

1.4 Error-correcting codes

The following 8 bitstrings have the desired property: 0000000, 0010111, 0101101, 0111010, 1001011, 1011100, 1100110, 1110001.

Note that one can use the above bitstrings (of length 7) to encode bitstrings of length 3. To this end, one assigns to each possible bitstring of length 3 (that is, $000, 001, \ldots$) one of the 8 bitstrings of length 7 (called codewords). Moreover, note that, since the minimal distance between two codewords is 4, such encoding has the property that whenever one changes a codeword at at most 3 positions, this error can still be detected.

If you wonder, how the above 8 bitstrings have been found, you can read more about error-correcting codes. Some facts about such codes will be introduced in Chapter 5. There exists an efficient algorithm to generate bitstrings with the desired property, but it is beyond the scope of this lecture.

[2] This notation will be introduced in Chapter 4.5. In essence, $k \equiv_m l$ means that k and l yield the same remainder, when divided by m. Equivalently, $k \equiv_m l$ whenever m divides $l - k$.