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Diskrete Mathematik

Solution 4

Part 1: Proof techniques

4.1 Proof by Contradiction (2.4.8)

a) Let x be any irrational number and let r be any rational number. Assume that $s = x+r$ is rational. To reach a contradiction, we show that in such case x must be rational. Indeed, we have $x = s - r$. Therefore, we have that x is a difference of two rational numbers and thus, by the fact from the hint, it must also be rational. This is a contradiction with the assumption that x is irrational.

b) Assume for contradiction that $2^{\frac{1}{n}}$ is rational for some $n > 2$. That is, assume that there exist two positive integers, call them p and q, such that $2^{\frac{1}{n}} = \frac{p}{q}$. This implies that $2 = \frac{p^n}{q^n}$. Hence, we have $q^n + q^n = p^n$, which is a contradiction with Fermat's Last Theorem.

The contradiction with Fermat's Last Theorem follows from the counterexample $q^n + q^n = p^n$.

4.2 Existence Proof (2.4.9)

a) For $a = b = \sqrt{2}$, we have $ab = \sqrt{2}\sqrt{2} = 2$, which is rational. Since $\sqrt{2}$ is irrational (see Example 2.24), $a = b = \sqrt{2}$ is an example that proves the statement.

b) Let us again consider $a = b = \sqrt{2}$. If it is the case that $a^b = \sqrt{2}^{\sqrt{2}}$ is rational, the example $a = b = \sqrt{2}$ proves the statement. Otherwise, assume that $\sqrt{2}^{\sqrt{2}}$ is irrational. Consider $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, which are both irrational under this assumption. We have that $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$ is rational, which proves the statement.

This proof is not constructive, since the choice of the example a, b depends on whether $\sqrt{2}^{\sqrt{2}}$ is rational or not. In fact, it was first shown in 1973 that $\sqrt{2}^{\sqrt{2}}$ is irrational.

4.3 Pigeonhole Principle (2.4.10)

a) Let us consider the great circle [1] passing through two of the five points. There are two closed hemispheres, having this great circle as the border. Note that the two points

[1] A great circle of a sphere is the largest circle that can be drawn on this sphere.

lie on both of these hemispheres. By the pigeonhole principle, two of the remaining three points must lie on the same hemisphere. Thus, this hemisphere must contain four points (together with the two on the great circle).

b) For every day i of November ($1 \leq i \leq 30$), let us consider the number a_i of bananas eaten by the monkey until that day (together with the day i). That is, on the first day it ate a_1 bananas, during the first two days it ate a_2, and so on. Further, let $b_i = a_i + 14$ for $1 \leq i \leq 30$.

First, note that for each $i \in \{1, \ldots, 30\}$, it holds that $1 \leq a_i < b_i \leq 59$ (the last inequality follows from the fact that the monkey had only 45 bananas and $45 + 14 = 59$). Hence, we have 60 numbers $a_1, \ldots, a_{30}, b_1, \ldots, b_{30}$, all between 1 and 59. By the pigeonhole principle, at least two of these numbers must be equal.

Notice now that we have $a_1 < a_2 < \cdots < a_{30}$, since the monkey ate at least one banana every day. By the definition of b_i, the same must hold for the sequence $b_1, \ldots, b_{30}$, that is $b_1 < b_2 < \cdots < b_{30}$. Therefore, the two equal numbers must be a_i and b_j for some i, j. Note further that we must have $i > j$. Otherwise, we would have $a_i = b_j$ for $i \leq j$. But since $b_j > a_j$, it would follow that $a_i > a_j$ for $i \leq j$, which is the contradiction with the fact that $a_1 < a_2 < \cdots < a_{30}$.

Thus, we have $a_i = b_j$ for some $j < i$. It follows that $a_i = 14 + a_j$ and, hence, $a_i - a_j = 14$. The value $a_i - a_j$ is exactly the amount of bananas the monkey ate between days j and i (including day i and excluding day j).

4.4 Proof by Induction (2.4.12)

Basis: For $n = 0$, one can simply use one color to color the whole plane.

Induction step: Fix any $n \geq 0$ and assume that the statement holds for this n. That is, assume that for any n lines it is possible to color the regions, using only two colors.

Let us consider any $n + 1$ lines on a plane. We show a procedure to color the regions on this plane. To this end, we first remove one of the lines (call this line l). Then there are only n lines left and we can color the regions with two colors by the induction hypothesis. Afterwards, l is placed back on the plane. Then, we pick one side of l and call it A, while the other side we call B (note that now with $n + 1$ lines, there are more regions than with only n lines). For every region on side A, we swap its color. Side B is left unchanged.

Let us now argue why the resulting coloring is correct. What we need to show is that for each border, the regions on both sides of it have different colors (note that, of course, no border can cross l). By the induction hypothesis, this condition is trivially satisfied for the borders on side B. Moreover, it is also satisfied for borders on side A,since swapping the color does not affect the correctness of coloring. What is left is to argue about new borders, lying on the line l. But such borders also cannot violate the condition, since before l was removed the two regions with border on l were part of the same region, so they had one color. After we swapped the color on side A, the colors must have become different.

Part 2: Set Theory

4.5 Operations on sets and cardinality

a) The following sets fulfill the conditions:

i) $A = \{\varnothing\}$
For $x = \varnothing$ we have $x \in A$. Also, the empty set is the subset of any other set. Hence, $x \subseteq A$. *(1 Point)*
This is not the only solution. For example, $A = \{7, \{7\}\}$ also fulfills the given condition.

ii) $A = \{\varnothing, 1\}$
We have $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}, \{1\}, \{\varnothing, 1\}\}$. Since $1 \notin \mathcal{P}(A)$, it holds that $A \not\subseteq \mathcal{P}(A)$. Also, for $x = \varnothing$ we have $x \in A$ and $x \subseteq \mathcal{P}(A)$ (since the empty set is the subset of any set). *(1 Point)*

iii) $A = \varnothing$
We have $\varnothing \subseteq \mathcal{P}(A)$. The second requirement is trivially fulfilled, since A has no elements. *(1 Point)*

b) First, notice that $A = \{\varnothing, \{\varnothing\}\}$. With that said, we give the solutions to individual subtasks:

i) $A \cup B = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$, $|A \cup B| = 4$ *(1 Point)*

ii) $A \cap B = \{\{\varnothing\}\}$, $|A \cap B| = 1$ *(1 Point)*

iii) $\varnothing \times A = \varnothing$, $|\varnothing \times A| = 0$ *(1 Point)*

iv) $\{0\} \times \{3, 1\} = \{(0, 3), (0, 1)\}$, $|\{0\} \times \{3, 1\}| = 2$ *(1 Point)*

v) $\{\{1, 2\}\} \times \{3\} = \{(\{1, 2\}, 3)\}$, $|\{\{1, 2\}\} \times \{3\}| = 1$ *(1 Point)*

vi) $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$, $|\mathcal{P}(\{\varnothing\})| = 2$ *(1 Point)*

4.6 Proofs in set theory

a) We will prove the implication in both directions separately.

$A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$: Let B be any set and let A be any subset of B. What we have to show is that each element of $\mathcal{P}(A)$ is also an element of $\mathcal{P}(B)$. Let S be any element of $\mathcal{P}(A)$. Then, by Definition 3.5, $S \subseteq A$. By the assumption that $A \subseteq B$ and by the transitivity of $\subseteq$, it follows that $S \subseteq B$. This means that S is an element of $\mathcal{P}(B)$.

$\mathcal{P}(A) \subseteq \mathcal{P}(B) \Rightarrow A \subseteq B$: Let A, B be any sets and assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$ (which holds for any set A) and, by assumption, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have that $A \in \mathcal{P}(B)$. By Definition 3.5, this means that $A \subseteq B$.

b) For any x and y, we have

$$
\begin{aligned}
& (x,y) \in A \times (B \cap C) & \\
\Leftrightarrow\ & x \in A \land y \in (B \cap C) & \text{(definition of } \times\text{)} \\
\Leftrightarrow\ & x \in A \land (y \in B \land y \in C) & \text{(definition of } \cap\text{)} \\
\Leftrightarrow\ & (x \in A \land x \in A) \land (y \in B \land y \in C) & \text{(idempotence of } \land\text{)} \\
\Leftrightarrow\ & x \in A \land x \in A \land y \in B \land y \in C & \text{(associativity of } \land\text{)} \\
\Leftrightarrow\ & x \in A \land y \in B \land x \in A \land y \in C & \text{(commutativity of } \land\text{)} \\
\Leftrightarrow\ & (x \in A \land y \in B) \land (x \in A \land y \in C) & \text{(associativity of } \land\text{)} \\
\Leftrightarrow\ & (x,y) \in A \times B \land (x,y) \in A \times C & \text{(definition of } \times\text{)} \\
\Leftrightarrow\ & (x,y) \in (A \times B) \cap (A \times C) & \text{(definition of } \cap\text{)}
\end{aligned}
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