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Diskrete Mathematik

Solution 14

14.1 Statements about formulas

a) This expression is a syntactically correct formula.

b) This is a statement about the formulas $\forall x\ P(x)$ and $P(x)$. It is true. Let $\mathcal{A}$ be an interpretation suitable for both $\forall x\ P(x)$ and $P(x)$, which is a model for $\forall x\ P(x)$. Since $\mathcal{A}(\forall x\ P(x)) = 1$, it follows that for all $u \in U^{\mathcal{A}}$, $\mathcal{A}_{[x \to u]}(P(x)) = 1$. Hence, no matter which $x^{\mathcal{A}} \in U^{\mathcal{A}}$ is assigned to x by $\mathcal{A}$, $\mathcal{A}(P(x)) = 1$. Therefore, $\mathcal{A}$ is also a model for $P(x)$.

c) This is not a valid expression, since $\equiv$ can only be used between formulas and $P(x) \models P(x)$ is a statement, not a formula.

d) This is a statement about formulas and it is false. As a counterexample, consider the structure: $U^{\mathcal{A}} = \{0, 1\}, P^{\mathcal{A}}(x) = 1 \iff x = 1, x^{\mathcal{A}} = 1, f^{\mathcal{A}}(x) \equiv 1, a^{\mathcal{A}} = 0$. Then we have $\mathcal{A}(P(x)) = 1$ and $\mathcal{A}(P(f(a))) = 1$, but $\mathcal{A}(P(a)) = 0$.

14.2 Relation between validity of a formula and statement about formulas

a) $F := (G_1 \wedge \cdots \wedge G_k) \to H$.

F is a tautology if and only if $\mathcal{A}((G_1 \wedge \cdots \wedge G_k) \to H) = 1$ for all suitable interpretations $\mathcal{A}$. This means that if $\mathcal{A}(G_1 \wedge \cdots \wedge G_k) = 1$, then we must have $\mathcal{A}(H) = 1$ as well. That is, for any interpretation $\mathcal{A}$, such that $\mathcal{A}(G_1) = \cdots = \mathcal{A}(G_k) = 1$, we have $\mathcal{A}(H) = 1$. This means exactly that $\{G_1, \ldots, G_k\} \models H$.

Note: This is only an example, any formula equivalent to F would also be a good solution.

b) $F := G \leftrightarrow H$.

F is a tautology if and only if $\mathcal{A}(G \leftrightarrow H) = 1$ for all suitable interpretations $\mathcal{A}$. This means that if $\mathcal{A}(G) = 1$, then $\mathcal{A}(H) = 1$ as well, and if $\mathcal{A}(H) = 1$, then $\mathcal{A}(G) = 1$ as well. Hence, by the definition of $\models$, we have $G \models H$ and $H \models G$. Therefore, $G \equiv H$.

Note: Again, this is only an example.

14.3 Calculi

a) The following rules are correct: R_1, R_2, R_4 and R_6.

To show this, for each rule R we consider the statement $M \models H$ for a set M and a formula H. If this statement is true for any M and H such that $M \vdash_R H$, then the rule is correct. We show $M \models H$ by drawing a function table and checking that the truth value of H is 1 whenever the truth values of all formulas in M are 1. A rule is incorrect if the statement $M \models H$ is false. We show this by giving a counterexample (the counterexamples are the rows in the corresponding function tables, printed in bold).

	F	G	F	$F \vee G$
	0	0	0	0
R_1:	0	1	0	1
	1	0	1	1
	1	1	1	1

	F	G	$F \wedge G$	F
	0	0	0	0
R_2:	0	1	0	0
	1	0	0	1
	1	1	1	1

	F	G	$\neg(F \wedge G)$	$\neg F \wedge \neg G$
	0	0	1	1
R_3:	**0**	**1**	**1**	**0**
	1	**0**	**1**	**0**
	1	1	0	0

	F	G	F	$F \to G$	G
	0	0	0	1	0
R_4:	0	1	0	1	1
	1	0	1	0	0
	1	1	1	1	1

	F	G	$F \to G$	$\neg F \to \neg G$
	0	0	1	1
R_5:	**0**	**1**	**1**	**0**
	1	0	0	1
	1	1	1	1

	F	G	$F \wedge G$
	0	0	0
R_6:	0	1	0
	1	0	0
	1	1	1

b) We have $K = \{R_1, R_2, R_4, R_6\}$. The derivation is the following:

$$\{B \wedge A\} \vdash_{R_2} B$$
$$\{B\} \vdash_{R_1} B \vee C$$
$$\{B \vee C, (B \vee C) \to D\} \vdash_{R_4} D$$
$$\{A \wedge B\} \vdash_{R_2} A$$
$$\{D, A\} \vdash_{R_6} D \wedge A$$
$$\{D \wedge A, (D \wedge A) \to C\} \vdash_{R_4} C$$
$$\{A \wedge B, C\} \vdash_{R_6} A \wedge B \wedge C$$
$$\{A \wedge B \wedge C, D\} \vdash_{R_6} A \wedge B \wedge C \wedge D$$

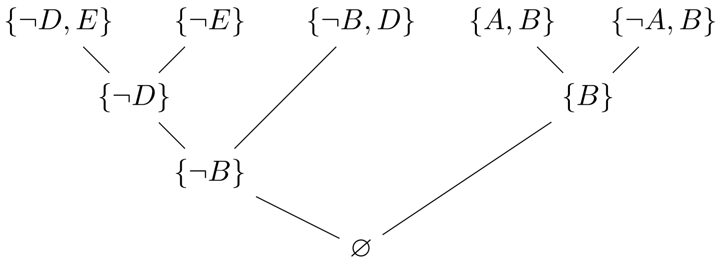
c) The calculus $K = \{R_2, R_4\}$ is not complete. As a counterexample, consider the set $M_0 = \{A \wedge B\}$ and the formula $H := B \wedge A$. We have $A \wedge B \models B \wedge A$. However, H cannot be derived from M_0. Indeed, to M_0 one can only apply R_2 with $F := A$ and $G := B$, obtaining the set $M_1 = \{A \wedge B, A\}$. But no new formulas can be derived from M_1.

d) For example, the following calculus $K := \{R\}$ with $\varnothing \vdash_R F$ is complete but not sound.

In the calculus K, one can derive exactly *all* formulas. Hence, it is clearly complete. It is also clearly not sound, since for example, the formula $A \wedge B$ can be derived and it is not a tautology.

14.4 Resolution in propositional logic

a) **i)** The clauses are: $\{A, B\}, \{\neg E\}, \{\neg B, D\}, \{\neg D, E\}, \{\neg A, B\}$.

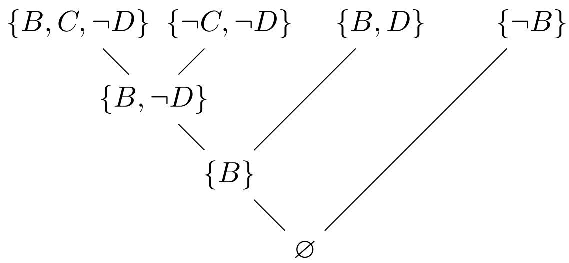


Hence, the formula is not satisfiable.

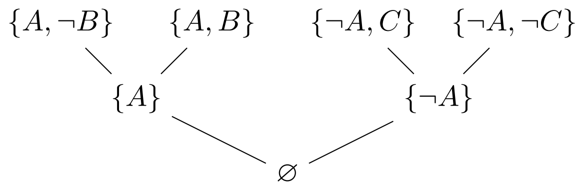
ii) The formula $G = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$ is a tautology if and only if

$$\neg G \equiv (B \vee C \vee \neg D) \wedge (B \vee D) \wedge (\neg C \vee \neg D) \wedge (\neg B)$$

is not satisfiable. We show this, using the resolution calculus:



iii) Let $\mathcal{K}(M) = \{\{\neg A, C\}, \{A, \neg B\}, \{A, B\}\}$ be the set of clauses, corresponding to the set M. The set of clauses corresponding to $\neg H$ is $\mathcal{K}(\neg H) = \{\neg A, \neg C\}$ We show that $\mathcal{K}(M) \cup \mathcal{K}(\neg H)$ is unsatisfiable.

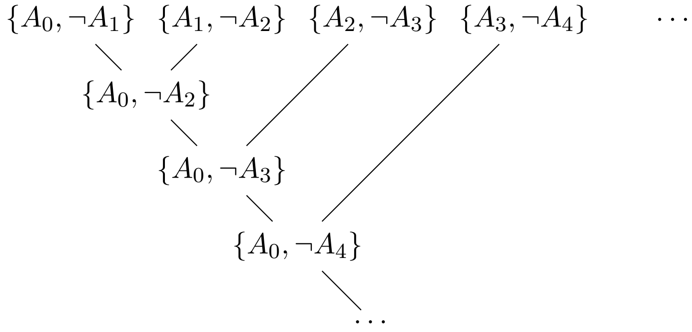


b) There is only a finite number of atomic formulas in $\mathcal{K}$. Let k denote their number. Since in a clause an atomic formula can either: appear plain, appear negated, appear in both forms or not appear at all, the number of possible clauses that can be derived from $\mathcal{K}$ is 4^k. Now for all $i \geq 0$, we have $\mathcal{K}_i \subseteq \mathcal{K}_{i+1}$. It follows that $|\mathcal{K}_i| \leq |\mathcal{K}_{i+1}|$, which, together with the fact that $|\mathcal{K}_i| \leq 4^k$, implies that for some $n \geq 0$, we have $|\mathcal{K}_n| = |\mathcal{K}_{n+1}| = \ldots$ It follows that no new clauses can be added, that is, $\mathcal{K}_n = \mathcal{K}_{n+1} = \ldots$.

c) For $i \in \mathbb{N}$, let

$$\mathcal{K}_i := \mathcal{K} \cup \bigcup_{j=1}^{i} \{\{A_0, \neg A_{j+1}\}\}.$$

Graphically, the constructed sequence of derivations looks as follows:



More formally, we clearly have $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i \neq \mathcal{K}_{i-1}$ for all $i > 0$. What is left to show is that for all $i > 0$, there exist $K', K'' \in \mathcal{K}_{i-1}$ and K, such that $\{K', K''\} \vdash_{\mathsf{res}} K$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ (where K is the new clause, $K \notin \mathcal{K}_{i-1}$). Indeed, for any $i > 0$, we can take $K' = \{A_0, \neg A_i\} \in \mathcal{K}_{i-1}$ and $K'' = \{A_i, \neg A_{i+1}\} \in \mathcal{K} \subseteq \mathcal{K}_{i-1}$. Then we have $\{K', K''\} \vdash_{\mathsf{res}} \{A_0, \neg A_{i+1}\}$ (so $K = \{A_0, \neg A_{i+1}\}$) and

$$\mathcal{K}_i := \mathcal{K} \cup \bigcup_{j=1}^{i} \big\{\{A_0, \neg A_{j+1}\}\big\} = \mathcal{K} \cup \bigcup_{j=1}^{i-1} \big\{\{A_0, \neg A_{j+1}\}\big\} \cup \big\{\{A_0, \neg A_{i+1}\}\big\} = \mathcal{K}_{i-1} \cup \{K\}.$$