ETH Zürich, D-INFK
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# Diskrete Mathematik

# Exercise 3

## Part 1: Predicate logic

### 3.1 Quantifiers and predicates *(10 Points)*

In this exercise the universe is fixed to the set $\mathbb{Z}$ of integers.

**a)** For each of the following statements, give a formula where the only predicates are `less`, `equals` and `prime` (instead of `less`$(n, m)$ and `equals`$(n, m)$ you can write $n < m$ and $n = m$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote addition and multiplication.

**i)** ($\star\star$) If the product of two integers is positive, then at least one of these numbers is positive. *(2 Points)*

**ii)** ($\star\star$) For every natural number, one can find a strictly greater natural number that is divisible by 3. *(2 Points)*

**iii)** ($\star\star\star$) Every even integer greater than 2 is the sum of two primes. *(2 Points)*

Bonus question: Which of the above statements are true?

**b)** Consider the following predicates $P(x)$ and $Q(x, y)$:

$$P(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases} \qquad Q(x,y) = \begin{cases} 1, & xy = 1 \\ 0, & \text{otherwise} \end{cases}$$

In this context, describe the following formulas as mathematical statements (that is, write them using words). Also, for each of the formulas decide whether it is true or false.

**i)** ($\star\star$) $\forall x\, \exists y\, Q(x, y)$ *(2 Points)*

**ii)** ($\star\star$) $\exists x \big(\forall y\, \neg Q(x, y) \;\wedge\; \exists y\, P(y)\big)$ *(2 Points)*

### 3.2 Transitivity of quantifiers

Prove that:

**a)** ($\star\star$) $\exists y\, \forall x\, P(x, y) \;\models\; \forall x\, \exists y\, P(x, y)$.

Disprove that:

**b)** ($\star\star$) $\forall x\, \exists y\, P(x, y) \;\models\; \exists y\, \forall x\, P(x, y)$.

### 3.3 Winning strategy ($\star\star$)

Alice and Bob play a game in which the stake is a chocolate bear. Rules of the game are the following: Alice chooses two integers $a_1, a_2$ and Bob chooses two integers $b_1, b_2$. Alice wins whenever $a_1 + (a_2 + b_1)^{|b_2|+1} = 1$ and Bob wins otherwise.

**a)** First, consider the case when Alice and Bob announce all their numbers at the same time. Give a formula that describes the statement "Alice has a winning strategy." Is this statement true?

**b)** In the second case, Alice and Bob declare their numbers one by one. That is, first Alice announces $a_1$, then Bob announces $b_1$, afterwards Alice announces $a_2$, and at the end Bob replies with $b_2$. Once again, give a formula that describes the statement "Alice has a winning strategy.". Is this statement true in this case?

## Part 2: Proof techniques

### 3.4 Direct Proof of an Implication (2.4.3)

Prove directly:

**a)** ($\star$) The product of two even natural numbers is even.

### 3.5 Indirect Proof of an Implication (2.4.4)

Prove indirectly that for all natural numbers $n > 0$, we have:

**a)** ($\star\star$) If $42^n - 1$ is a prime, then $n$ odd.

**b)** ($\star\star$) If $n^2$ is odd, then $n$ is also odd.

### 3.6 Case Distinction (2.4.7)

Prove by case distinction that:

**a)** ($\star\star$) For all integers $n$, we have that $5n^2 + 3n + 42$ is even.

**b)** ($\star\star\star$) If $p$ and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.

**Due on 9. October 2017.**
**Exercise 3.1. will be corrected.**