ETH Zürich, D-INFK
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# Diskrete Mathematik

# Exercise 4

## Part 1: Proof techniques

### 4.1 Proof by Contradiction (2.4.8)

**a)** ($\star\star$) Show by contradiction that the sum of a rational number and an irrational number is irrational.

*Hint: Use the fact that the difference of two rational numbers is rational.*

**b)** ($\star\star\star$) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

*Hint: Fermat's Last Theorem states that no positive integers $a, b, c$ satisfy the equation $a^n + b^n = c^n$ for $n > 2$.*

### 4.2 Existence Proof (2.4.9)

Prove that:

**a)** ($\star$) There exist irrational numbers $a$ and $b$ such that $ab$ is rational.

**b)** ($\star\star\star\star$) There exist irrational numbers $a$ and $b$ such that $a^b$ is rational.

### 4.3 Pigeonhole Principle (2.4.10)

**a)** ($\star\star$) Given are five points on a sphere. Show that there exists a closed hemisphere (i.e., a hemisphere that includes its boundary) which contains four of these points.

**b)** ($\star\star\star$) On the $1^{\text{st}}$ of November, a monkey bought $45$ bananas. On each of the following $30$ days it ate a certain number of its bananas. It had to eat at least one banana every day, in order not to starve. Prove that there must exist a period of consecutive days in which the monkey ate exactly $14$ bananas.

### 4.4 Proof by Induction (2.4.12) ($\star\star\star$)

Consider a scenario, where we draw $n$ straight lines on a plane. These lines divide the plane into a number of regions. Prove that for any $n \geq 0$, it is possible to color these regions using only two colors, such that no two adjecent regions (i.e., regions that share a border) have the same color. Two regions that only share a point are not adjecent.

## Part 2: Set Theory

### 4.5 Operations on sets and cardinality ($\star\star$) *(9 Points)*

**a)** In each of the following cases, give a set $A$ such that

**i)** there exists an $x \in A$ such that $x \subseteq A$.

**ii)** $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$. *(3 Points)*

**iii)** $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

**b)** Let $A = \{\varnothing, \{\varnothing\}, \{\varnothing\}\}$ and $B = \{A, \{\varnothing\}, \{\{\varnothing\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality. *(6 Points)*

**i)** $A \cup B$ **ii)** $A \cap B$ **iii)** $\varnothing \times A$

**iv)** $\{0\} \times \{3, 1\}$ **v)** $\{\{1, 2\}\} \times \{3\}$ **vi)** $\mathcal{P}(\{\varnothing\})$

### 4.6 Proofs in set theory

**a)** ($\star\star$) Let $A$ and $B$ be arbitrary sets. Prove that $A \subseteq B \Longleftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$.

**b)** ($\star\star$) Let $A$ and $B$ be arbitrary sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

**Due on 16. October 2017.**
**Exercise 4.5. will be corrected.**