ETH Zürich, D-INFK
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Diskrete Mathematik

Exercise 12

Part 1: Propositional logic

12.1 Syntax and semantic of XOR ($\star\star$)

We would like to extend propositional logic by the symbol $\oplus$, denoting the exclusive or (XOR) operation ($A \oplus B$ is true if and only if either A or B is true, but not both). How would one extend the definitions of syntax and semantics of propositional logic, in order to incorporate XOR?

12.2 Satisfiability ($\star\star$)

a) Prove or disprove the following statement: if F and G are formulas of propositional logic such that F and $F \to G$ are satisfiable, then G is also satisfiable.

b) For each of the following sets of formulas, either find a model or show that it is unsatisfiable.

i. $M = \{\neg A,\ B \wedge C,\ \neg A \to \neg C\}$

ii. $N = \{A_1 \vee A_2,\ \neg A_2 \vee A_3,\ \neg A_3 \vee A_4, \ldots\}$

12.3 Normal forms ($\star$)

a) Let $F := (\neg A \to B \wedge C) \leftrightarrow \neg C$. Using the method of function tables, construct a formula equivalent to F in conjunctive normal form and a formula equivalent to F in disjunctive normal form.

b) Let $G := (A \wedge \neg B) \vee (\neg A \wedge (C \wedge D))$. Using the equivalences from Lemma 6.2, construct a formula equivalent to G in conjunctive normal form. In each step write which equivalence you use.

Part 2: Predicate logic

12.4 Structures and models ($\star$) *(6 Points)*

a) Which of the following suitable structures i), ii) and iii) are models for the following formula? Justify your answers.

$$F := \forall x \forall y \forall z \left(P(f(x,y),x) \ \wedge\ P(f(x,y),y) \ \wedge\ \left(\neg P(x,y) \to \neg P(x, f(y,z))\right)\right)$$

i) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}$, $f^{\mathcal{A}}(x,y) = x \cdot y$, $P^{\mathcal{A}}(x,y) = 1 \;:\Longleftrightarrow\; y \mid x$
ii) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}$, $f^{\mathcal{A}}(x,y) = x^y$, $P^{\mathcal{A}}(x,y) = 1 \;:\Longleftrightarrow\; y \mid x$
iii) $U^{\mathcal{A}} = \mathcal{P}(\mathbb{N})$, $f^{\mathcal{A}}(A,B) = A \cap B$, $P^{\mathcal{A}}(A,B) = 1 \;:\Longleftrightarrow\; A \subseteq B$

The symbol | denotes the divisibility relation. *(3 Points)*

b) For the formula $G := \forall x \exists y \left(P(x,y) \wedge \neg P(f(x),y) \wedge Q(y,z)\right)$ give a structure, which is

i) suitable and a model for G.

ii) suitable but not a model for G.

iii) not suitable for G. *(3 Points)*

12.5 Free variables (⋆)

Determine all free variables in the following formulas:

i) $\forall x \forall y \, (P(x,y) \vee P(x,z))$

ii) $\forall x \, (\exists x \, P(x) \wedge P(x)) \vee P(x)$

iii) $\forall x \, \left(\exists y \, P(y,x) \vee \exists z \, Q(x, f(z))\right)$

12.6 Tautologies (⋆ ⋆)

Let F be a formula, and let $x_1, ..., x_n$ be all the variables that occur free in F. Show that F is valid if and only if $\forall x_1 \ldots \forall x_n \, F$ is valid.

Due on 11. December 2017.
Exercise 12.4 will be corrected.