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Diskrete Mathematik

Exercise 1

This exercise sheet contains some mathematical riddles and brain teasers, which are, in one way or another, connected to the topics of this lecture. The solutions will be discussed during the first exercise class. You are not expected to be able to fully solve all the exercises.

1.1 Hilbert's Hotel

David Hilbert [1] is the manager of a hotel with an infinite number of rooms, numbered by $1, 2, 3, \ldots$, all of which are occupied. Each guest is willing to change his or her room at most once.

a) ($\star\,\star$) Roger Federer comes to the hotel and asks whether there is a free room for him. Hilbert cannot turn away the distinguished guest and promises him a room. To make it possible, he tells some of the other guests to change their rooms. How can he do this, so that at the end every current guest still has a room?

b) ($\star\,\star\,\star$) One day, at the door of the hotel arrives a bus with an infinite number of passengers, numbered by $1, 2, 3, \ldots$, each of whom would like a single room. How can Hilbert accommodate all newly arrived guests, so that all current guests still have a place to stay?

c) ($\star\,\star\,\star\,\star$) An infinite number of buses, numbered by $1, 2, 3, \ldots$, each carrying an infinite number of passengers, arrives at the hotel. How can Hilbert deal with this situation?

1.2 Alice in the Forest of Forgetfulness ($\star\,\star$)

When Alice [2] entered the Forest of Forgetfulness, she forgot many things. For example, she cannot remember her name and the day of the week. She knows, however, that there are two strange brothers living in the forest: Tweedledum and Tweedledee. The first one always lies on Mondays, Tuesdays and Wednesdays and tells the truth on the other days of the week. The other lies on Thursdays, Fridays and Saturdays and tells the truth on the other days. From the outside, they look much alike and Alice cannot tell them apart. One day, Alice meets both brothers together. One of them says „I'm Tweedledum!", while the other claims "I'm Tweedledee!". Which one is really Tweedledum?

[1] David Hilbert (1862 – 1943) was a prominent mathematician, who in fact proposed a thought experiment, concerning a hotel with an infinite number of rooms.

[2] This is a riddle from the book „What Is the Name of This Book?: The Riddle of Dracula and Other Logical Puzzles" by Raymond M. Smullyan. If you would like to practice logic or simply enjoy solving such puzzles, this book is a great source of similar riddles.

1.3 The Island ($\star\,\star$)

On a tropical island, seven pirates and one monkey collect coconuts. There are at most 250 coconuts on the whole island. After a full day of hard work, the pirates and the monkey want to share the coconuts among themselves. If they divided the coconuts into 8 equal parts, they would be left with one extra coconut. Since no pirate wants to allow anyone else to have one additional coconut, they decide to instead divide the coconuts into 7 equal parts. In such case, 2 extra coconuts are left, which can be given to the monkey. However, one of the pirates disagrees, claiming that this would be unfair for the monkey. This leads to a duel between him and another pirate. The fight turns out to be fatal for both of them. In the end, the remaining pirates divide the coconuts into 5 equal parts and give the 3 remaining ones to the monkey.

How many coconuts did each surviving pirate get?

1.4 The Punctured Chessboard ($\star$)

In the lecture (see also Example 1.1 in the lecture notes) we considered a $k \times k$ chessboard with one of the squares punctured. We also defined the predicate $P(k)$ to be equal 1 whenever the following statement is true:

> *No matter which of the squares is punctured, the remaining area of the chessboard (consisting of $k^2 - 1$ squares) can be covered completely with (non-overlapping) L-shaped pieces of paper consisting of three squares.*

In this exercise we consider the proof by case distinction that $P(7) = 1$.

a) What is the smallest number of cases one has to consider in the proof? (Consider symmetries of the chessboard.)

b) Carry out the proof for two of the cases.