

Diskrete Mathematik

Exercise 4

This exercise is not graded.

Part 1: Proof techniques

4.1 Proof by Contradiction (2.4.8)

- a) (**) Show by contradiction that the sum of a rational number and an irrational number is irrational.

Hint: Use the fact that the difference of two rational numbers is rational.

- b) (***) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

Hint: Fermat's Last Theorem states that no positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for $n > 2$.

4.2 Existence Proof (2.4.9)

Prove that:

- a) (*) There exist irrational numbers a and b such that ab is rational.
b) (***) There exist irrational numbers a and b such that a^b is rational.

4.3 Pigeonhole Principle (2.4.10)

- a) (**) Given are five points on a sphere. Show that there exists a closed hemisphere (i.e., a hemisphere that includes its boundary) which contains four of these points.
b) (***) On the 1st of November, a monkey bought 45 bananas. On each of the following 30 days it ate a certain number of its bananas. It had to eat at least one banana every day, in order not to starve. Prove that there must exist a period of consecutive days in which the monkey ate exactly 14 bananas.

4.4 Proof by Induction (2.4.12) (***)

Consider a scenario, where we draw n straight lines on a plane. These lines divide the plane into a number of regions. Prove that for any $n \geq 0$, it is possible to color these regions using

only two colors, such that no two adjacent regions (i.e., regions that share a border) have the same color. Two regions that only share a point are not adjacent.

Part 2: Set Theory

4.5 Operations on sets and cardinality (★ ★) (9 Points)

- a) In each of the following cases, give a set A such that
- i) there exists an $x \in A$ such that $x \subseteq A$.
 - ii) $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$. (3 Points)
 - iii) $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.
- b) Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset\}\}$ and $B = \{A, \{\emptyset\}, \{\{\emptyset\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.
- i) $A \cup B$
 - ii) $A \cap B$
 - iii) $\emptyset \times A$
 - iv) $\{0\} \times \{3, 1\}$
 - v) $\{\{1, 2\}\} \times \{3\}$
 - vi) $\mathcal{P}(\{\emptyset\})$
- (6 Points)

4.6 Proofs in set theory

- a) (★ ★) Let A and B be arbitrary sets. Prove that $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- b) (★ ★) Let A and B be arbitrary sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
- c) (★ ★) Let $n > 0$ be a natural number, and let A_1, \dots, A_n be sets. Prove that

$$\overline{A_1 \cup \dots \cup A_n} = \overline{A_1} \cap \dots \cap \overline{A_n}.$$

Due on 16. October 2018.
Exercise 4.5. will be corrected.