# Anonymous Symmetric-Key Communication*

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**Abstract.** We study anonymity of *probabilistic encryption* (pE) and *probabilistic authenticated encryption* (pAE). We start by providing concise game-based security definitions capturing anonymity for both pE and pAE, and then show that the commonly used notion of *indistinguishability from random ciphertexts* (IND$) indeed implies the anonymity notions for both pE and pAE. This is in contrast to a recent work of Chan and Rogaway (Asiacrypt 2019), where it is shown that IND$-secure nonce-based authenticated encryption can only achieve anonymity if a sophisticated transformation is applied. Moreover, we also show that the *Encrypt-then-MAC* paradigm is anonymity-preserving, in the sense that if both the underlying probabilistic MAC (pMAC) and pE schemes are anonymous, then also the resulting pAE scheme is. Finally, we provide a composable treatment of anonymity using the constructive cryptography framework of Maurer and Renner (ICS 2011). We introduce adequate abstractions modeling various kinds of anonymous communication channels for many senders and one receiver in the presence of an active man-in-the-middle adversary. Then we show that the game-based notions indeed are anonymity-preserving, in the sense that they imply constructions between such anonymous channels, thus generating authenticity and/or confidentiality as expected, but crucially retaining anonymity if present.

**Keywords:** anonymous encryption · anonymous authenticated encryption · composable security · composable anonymity · anonymous channel

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# 1 Introduction

When transmitting messages in the symmetric-key setting, where communicating parties share secret keys a priori, traditionally *confidentiality* and *authenticity* are the security properties that are mostly considered. Confidentiality guarantees exclusivity of the receiving party (no one but the receiver should be able to gain any partial information about the transmitted message, possibly other than its length), while authenticity guarantees exclusivity of the sending party (no one except the sender should be able to convince the receiver that it indeed originated the message). But in a scenario where there are more than just two communicating parties using the same protocol, e.g., many senders and one receiver (as considered in this work), another important security property must be taken into account, namely *anonymity*.

For the mentioned setting, we are more specifically interested in *external sender anonymity*, that is, the property that guarantees that no one but the receiver can learn from which sender a message originated. The main focus of our work is on security definitions which capture exactly this guarantee (in particular, note that we are not addressing other common forms of anonymity usually found in the literature, arising for instance from traffic-flow analysis).

## 1.1 Background

Anonymity, as opposed to confidentiality and authenticity, in most settings (as is the case for the one considered here) cannot be "created out of the blue"; rather, an intrinsic property of anonymity is that it can be *preserved*. In the game-based spirit of security definitions, this is reflected by the fact that conventional anonymity notions are captured by the concept of *key-indistinguishability* of a scheme originally intended to provide other forms of security, as confidentiality or authenticity. More specifically, in the symmetric-key setting this means that anonymity is a property that needs to be provided in conjunction with confidentiality for encryption schemes and with authenticity for MAC schemes.

But when considered from a composable standpoint, the fact that anonymity can merely be preserved becomes even more evident: consider for example a protocol employing a MAC scheme and shared secret keys between the senders and the receiver, which is executed on top of an insecure channel to obtain an authenticated channel; if one wishes for the constructed channel to additionally be also anonymous, it must be the case that the insecure channel is anonymous as well, and this construction is still possible precisely if the employed MAC scheme not only is unforgeable, but is also key-indistinguishable.

The latter considerations were made explicit by Alwen, Hirt, Maurer, Patra, and Raykov in [AHM+15], and our work can be seen as a continuation and refinement of this line of research: Here we consider the construction of an anonymous *secure* (confidential *and* authenticated) channel from an anonymous authenticated one, and show that this is possible precisely if the employed encryption scheme not only has indistinguishable ciphertexts, but also indistinguishable keys. Moreover, we show that only if a secure authenticated encryption scheme which

is key-indistinguishable is employed, one can construct the anonymous secure channel directly from the anonymous insecure one.

### 1.2 Contributions

We consider the following setting: $n$ parties, the senders, wish to securely and anonymously transmit messages to the same party, the receiver, and we assume that the receiver a priori shares a (different) secret key with each of the $n$ senders. Since all of our treatment is in the *symmetric-key* setting, and the considered protocols employ *probabilistic* (as opposed to nonce-based) schemes, we often tacitly assume these two facts throughout the paper. Moreover, since the meaning of security usually depends on the context, we adopt the convention that for a cryptographic scheme by *anonymous security* we mean anonymity (in form of key-indistinguishability) in conjunction with its conventionally associated security notion, that is, confidentiality for encryption, authenticity for MAC, and confidentiality plus authenticity (usually simply referred to as just security) for authenticated encryption.

*Game-Based Security Definitions.* We start by providing game-based security definitions capturing anonymity for both *probabilistic encryption* (pE) and *probabilistic authenticated encryption* (pAE). For the former, we revisit the notion of *key-indistinguishability*, originally put forth by Fischlin [Fis99], and subsequently treated in [Des00] by Desai and in [AR00] by Abadi and Rogaway. In all three works this notion has been expressed for $n = 2$ senders; here we generalize it to an arbitrary number of senders. For *nonce-based* authenticated encryption (nAE), the analogous notion of key-indistinguishability has been recently put forth by Chan and Rogaway [CR19]. Here we propose a concise definition for the case of pAE instead.

For both pE and pAE we show the relevant implications among the introduced security definitions, exposing the concrete security losses surfacing from the reductions. Furthermore, we formally show that indeed the strong security notion of *indistinguishability from random ciphertexts* (dubbed IND$, and valid for both schemes) implies key-indistinguishability. Finally, we prove that the Encrypt-then-MAC (EtM) paradigm, applied on secure and anonymous pE and probabilistic MAC (pMAC), yields pAE which is not only secure, but crucially also anonymous, thus confirming that EtM is *anonymity-preserving*.

*Composable Security Definitions.* We next move to the focal point of our work, the composable treatment of anonymity. Here we introduce alternative security definitions within the *Constructive Cryptography* (CC) framework of Maurer and Renner [MR11,Mau12], which enjoy composability and allow to make explicit security goals from an application point of view.

First we phrase the desired security properties of (symmetric-key) protocols as specific constructions of cryptographic communication channels. More concretely, we start by defining the following resources which expose $n$ interfaces to send messages and one to receive them: the *insecure anonymous channel* (A-INS), the

*authenticated anonymous channel* (A-AUT), and the *secure anonymous channel* (A-SEC). Then we state that a protocol (executed by the senders and the receiver, which share secret keys a priori) provides *authenticity in conjunction with anonymity* if it constructs A-AUT from A-INS, provides *confidentiality in conjunction with anonymity* if it constructs A-SEC from A-AUT, and provides *security (i.e., confidentiality and authenticity) in conjunction with anonymity* if it constructs A-SEC directly from A-INS.

Secondly, we establish relations between the previously introduced game-based security definitions and their composable counterparts, that is, we show sufficiency conditions in terms of game-based definitions for the above mentioned constructions. As already mentioned earlier, in [AHM+15] it was shown that key-indistinguishable pMAC schemes enable the construction of A-AUT from A-INS. Here we show that anonymous secure pE enables the next logical step, namely the construction of A-SEC from A-AUT. In terms of time-complexity, this significantly improves upon the MAC-based solution proposed in [AHM+15] for the same construction. Furthermore, we show that these two steps can be performed in one shot using authenticated encryption instead, that is, we show that anonymous secure pAE constructs a A-SEC directly from A-INS. Again, this significantly improves upon the MAC-based solution proposed in [AHM+15] for the same construction. Moreover, this provides further evidence of the anonymity preservation of EtM.

*Preferring Probabilistic Schemes for Anonymity.* We observe that our constructive treatment strengthens the role of probabilistic authenticated encryption in contrast to its nonce-based counterpart when it comes to anonymity. According to Rogaway [Rog04], a main advantage provided by nonces is that

> *"encryption schemes constructed to be secure under nonce-based security notions may be less prone to misuse".*

Nevertheless, this raises concerns about attacks in the multi-user (mu) setting, where crucially anonymity lives. For this reason in TLS 1.3 a *randomized nonces* mechanism has been proposed for the employed nAE scheme, AES with GCM (Galois/Counter Mode). This recently spawned work by Bellare and Tackmann [BT16] and Hoang, Tessaro, and Thiruvengadam [HTT18], which initiated and refined the study of mu security of nAE in order to rigorously formalize security under such randomized nonces mechanism (but they did not address anonymity, in the form of key-indistinguishability).

But quoting again Rogaway [Rog11, I.8 (page 22)],

> "[if] *an IV-based encryption scheme* [...] *is good in the nonce-based framework* [...] *then it is also good in the probabilistic setting",*

which implies that an IND$-secure nAE scheme is an IND$-secure pAE scheme, when the nonce is randomized (if one ignores the concept of *associated data*). Therefore, in view of our previously mentioned result attesting that IND$-secure pAE implies anonymity, our work can be considered as a confirmation that the

random nonce mechanism, if used with an IND$-secure nAE scheme and under the assumption that the nonces are indeed truly uniformly random, also provides anonymity. Note that our consideration here is rather informal, and a more thorough study should be carried out to also incorporate the issue of nonce repetition and related birthday paradox security bounds (in our discussion, we are assuming a setting where not too many messages are exchanged).

This is to be compared to a recent work by Chan and Rogaway [CR19], which studies the anonymity of nAE: the authors observe that because of the session-related nature of the nonces, nAE actually fails to generally provide anonymity. For this reason, they introduce a transformation (dubbed NonceWrap) which converts an nAE scheme into a (syntactically different) new scheme, *anonymous* nAE (anAE), which they show does achieve anonymity (i.e., key-indistinguishability).

*A Framework for Security Definitions and Proofs.* We formulate all of the above mentioned security definitions in a systematic and concise language. We see the framework we put forth as an independent contribution, since it allows for compact formulations of security definitions, and enables easy and short (*reduction*-based) proofs of security, which in principle could be formally verified in a rather direct way (we leave this task open). Our proposed framework is based on the earlier work on *cryptographic systems* of Maurer, Pietrzak, and Renner [Mau02,MPR07], can be seen as a specialization of the recent work of Brzuska, Delignat-Lavaud, Fournet, Kohbrok, and Kohlweiss [BDLF+18], and is inspired by the approach taken by Rosulek in [Ros18].

### 1.3 Outline

We begin by providing the necessary background in Section 2, where we introduce our notation and the framework we use to state and prove security notions. As motivating examples, we revisit the classical security definitions for pE and pAE by capturing them within our framework. We proceed in Section 3 by providing game-based security definitions of anonymity, in terms of key-indistinguishability, for both pE and pAE. We introduce different notions, some capturing single security goals while others capturing more together, and then we show the relevant relations among them. Moreover, we show that for both pE and pAE, their respective stronger IND$ security notions imply anonymity. As a last result within the realm of game-based security notions, we show that the Encrypt-then-MAC paradigm, used to build secure pAE from secure pE and secure pMAC (whose syntax and security notions we introduce in Appendix C), not only preserves security, but anonymity as well. Finally, in Section 4 we provide composable security definitions capturing anonymity for both pE and pAE, and show that these notions are implied by the previously introduced game-based definitions. This is our main contribution, and it should be seen as shedding light into what anonymity (in the sense of key-indistinguishability) of symmetric cryptographic primitives really achieves from an application point of view. Our analysis makes it explicit that in this setting, key-indistinguishability must be understood as a tool that *preserves* anonymity, rather than creating it.

## 2 Preliminaries

We start by introducing some basic notation, and subsequently defining the building blocks of our framework for security proofs. We then provide syntax of $\mathsf{pE}$ and $\mathsf{pAE}$, and conclude the section by restating their (de facto standard) respective security definitions withing our framework.

### 2.1 Notation

We write $x, \ldots \leftarrow y$ to assign the value $y$ to variables $x, \ldots$, and $w, \ldots \xleftarrow{\text{iid}} \mathcal{D}$ to assign independently and identically distributed values to variables $w, \ldots$ according to distribution $\mathcal{D}$. $\varnothing$ denotes the empty set, $\mathbb{N} \doteq \{0, 1, 2, \ldots\}$ denotes the set of natural numbers, and for $n \in \mathbb{N}$, we use the convention $[n] \doteq \{1, \ldots, n\}$. For $n \in \mathbb{N}$, $\{0,1\}^n$ denotes the set of bitstrings of length $n$, $\{0,1\}^* \doteq \bigcup_{i \geq 0} \{0,1\}^i$ denotes the set of all finite length bitstrings, for $s \in \{0,1\}^*$, $|s|$ denotes the length of $s$ (in bits), and $\$^n$ represent a uniformly sampled random bitstring of length $n$. Finally, for a random variable $X$ over a set $\mathcal{X}$, $\operatorname{supp} X \doteq \{x \in \mathcal{X} \mid \Pr[X = x] > 0\}$.

### 2.2 Cryptographic Systems

We model cryptographic objects as *discrete reactive systems with interfaces*, that is, systems that can be queried with labeled inputs in a sequential fashion, where each distinct label corresponds to a distinct interface, and for each such input generate (possibly probabilistically) an equally labeled output depending on the input and the current state (formally defined by the sequence of all previous inputs and the associated outputs). Such systems can be formally described by conditional distributions of output values given input values, that is, by their *input-output behavior* (often described with *pseudocode*), as they formally correspond to *random systems* originally introduced in [Mau02], and later refined in [MPR07]. For two such systems $\mathbf{S}$ and $\mathbf{T}$ having the same input-output behavior (but possibly different implementation), we write $\mathbf{S} \equiv \mathbf{T}$.

In cryptography we are also interested in other objects (which can be formally modeled as special kinds of random systems). The first type we consider are *distinguishers*, which are just like the systems mentioned above, but enhanced with a special initial output which does not require an input, and a special final binary output. Formally, we usually consider a random experiment involving a distinguisher $\mathbf{D}$ and a system $\mathbf{S}$ which interact as follows: first $\mathbf{D}$ starts by (possibly probabilistically) generating the first output $X_1$ with some label (corresponding to a specific interface of $\mathbf{S}$), which will be used as the first input for $\mathbf{S}$ at that interface, which in turn will generate its first output $Y_1$ at the same interface, to be used as first input for $\mathbf{D}$. From $Y_1$ and the current state ($X_1$), $\mathbf{D}$ will then generate its second output $X_2$, with some (possibly different) label, and $\mathbf{S}$ will respond with $Y_2$ (depending on $X_1$, $Y_1$, and $X_2$), and so on, until $\mathbf{D}$ stops and outputs a bit $Z$. We call the operation of connecting $\mathbf{D}$ and $\mathbf{S}$ in the described way *sequential composition* and we syntactically represent it by the expression $\mathbf{DS}$, which is only valid if the number and types of labels

(interfaces) match.[1] We use the expression $\mathbf{DS}$ to also denote the random variable $Z$ representing $\mathbf{D}$'s final binary output.

The second type of special objects are *converters*, which are similar to systems but defining two disjoint sets of labels, and which can be used to extend either distinguishers (with labels matching the one in the first set) or systems (with labels matching the ones in the second set). We refrain from defining this concept on a formal level, and limit ourselves to give an intuitive description: a converter $\mathbf{C}$ is an object such that $\mathbf{DC}$ (the sequential composition restricted to the first set of labels of distinguisher $\mathbf{D}$ with $\mathbf{C}$) is again a distinguisher, and $\mathbf{CS}$ (the sequential composition restricted to the second set of labels of $\mathbf{C}$ with system $\mathbf{S}$) is again a system.[2]

As for example also done in [BDLF+18] and [Ros18], it is then possible to formalize an (associative) algebra of systems. Let $\mathbf{D}$ be a distinguisher, $\mathbf{C}$ a converter, and $\mathbf{S}$ a (regular) system. Then the experiment where $\mathbf{DC}$ interacts with $\mathbf{S}$ is the same experiment where $\mathbf{D}$ interacts with $\mathbf{CS}$, and we just denote this by $\mathbf{DCS}$ (again with the understanding that this expression also represents the final binary output of $\mathbf{D}$). Syntactically, this could be expressed as:[3]

$$(\mathbf{DC})\mathbf{S} = \mathbf{D}(\mathbf{CS}) = \mathbf{DCS}. \tag{1}$$

We next define another way to compose systems, *parallel composition*: given two (or more) systems $\mathbf{S}$ and $\mathbf{T}$, a new system $\mathbf{V}$ is the (independent) parallel composition of $\mathbf{S}$ and $\mathbf{T}$, denoted $\mathbf{V} = [\mathbf{S}, \mathbf{T}]$, if a system $\mathbf{D}$ interacting with $\mathbf{V}$ can (independently) access system $\mathbf{S}$ and system $\mathbf{T}$. We remark that $\mathbf{V}$ is merely a "wrapper" for two independent instances of systems $\mathbf{S}$ and $\mathbf{T}$. On the other hand, it is often also the case that two systems composed in parallel need some correlation, that is, need to lose their independence (usually trough a shared random variable or, more in general, some shared state); two such systems $\mathbf{S}$ and $\mathbf{T}$ might be used to create what is called a *correlated* parallel composition, which we formalize as a new system $\mathbf{V}$ such that $\mathbf{V} = \mathbf{C}[\mathbf{S}, \mathbf{T}]$, for some system $\mathbf{C}$ accessing the independent systems $\mathbf{S}$ and $\mathbf{T}$, and emulating two (correlated) systems towards a system $\mathbf{D}$ interacting with $\mathbf{V}$. We introduce the notation $\mathbf{V} = \langle \mathbf{S}, \mathbf{T} \rangle$, which makes the correlating system $\mathbf{C}$ implicit in the following sense: a system $\mathbf{D}$ interacting with $\mathbf{V}$ can access the system $\mathbf{S}$ and system $\mathbf{T}$, but only

[1] More formally, one could express $\mathbf{D}$ as $\mathbf{D}^i$, where $i \in \mathbb{N}$ is the number of different labels (interfaces) that $\mathbf{D}$ associates its output to, and analogously ${}^j\mathbf{S}$ for $\mathbf{S}$, where $j \in \mathbb{N}$; then a necessary condition for the sequential composition of $\mathbf{D}^i$ and ${}^j\mathbf{S}$ to be valid is that $i = j$, and in this case one would then write $\mathbf{D}^i\mathbf{S}$ instead of $\mathbf{DS}$.

[2] Again, one could express $\mathbf{D}$ as $\mathbf{D}^i$, for $i \in \mathbb{N}$, $\mathbf{C}$ as ${}^j\mathbf{C}^k$, for $j, k \in \mathbb{N}$ (where $j$ is the number of labels from the first set and $k$ from the second), and $\mathbf{S}$ as ${}^\ell\mathbf{S}$, for $\ell \in \mathbb{N}$; then a necessary condition for the sequential composition of $\mathbf{D}^i$ and ${}^j\mathbf{C}^k$ to be valid is that $i = j$, resulting in distinguisher $\mathbf{D}^i\mathbf{C}^k$, and a necessary condition for the sequential composition of ${}^j\mathbf{C}^k$ and ${}^\ell\mathbf{S}$ to be valid is that $k = \ell$, resulting in system ${}^j\mathbf{C}^k\mathbf{S}$.

[3] Using the more explicit notation, this would be: $(\mathbf{D}^i\mathbf{C}^j){}^j\mathbf{S} = \mathbf{D}^i({}^i\mathbf{C}^j\mathbf{S}) = \mathbf{D}^i\mathbf{C}^j\mathbf{S}$.

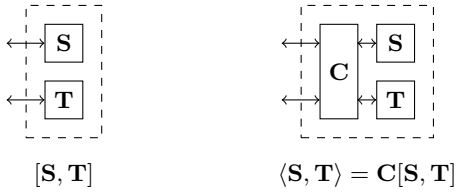


**Fig. 1:** Representation of the difference between *(independent) parallel composition* $[\mathbf{S}, \mathbf{T}]$ and *correlated parallel composition* $\langle \mathbf{S}, \mathbf{T} \rangle$.

through $\mathbf{C}$, and $\mathbf{S}$ and $\mathbf{T}$ become "labels"[4] for the correlated systems emulated by $\mathbf{C}$. Figure 1 illustrates the two different concepts. Note that we can naturally extend both definitions to the case of $n$ systems.

**Definition 1 (Systems Parallel Composition).** *Given the sequence of systems* $\mathbf{S}_1, \ldots, \mathbf{S}_n$*, for* $n \in \mathbb{N}$*, define:*

- *Their* (independent) parallel composition*, denoted* $[\mathbf{S}_1, \ldots, \mathbf{S}_n]$*, as the system that exports* $n$ *interfaces labeled* $\mathbf{S}_1, \ldots, \mathbf{S}_n$*, where label* $\mathbf{S}_i$ *is directly connected to system* $\mathbf{S}_i$*, for* $i \in [n]$*.*
- *Their* correlated parallel composition*, denoted* $\langle \mathbf{S}_1, \ldots, \mathbf{S}_n \rangle$*, as the system* $\mathbf{C}[\mathbf{S}_1, \ldots, \mathbf{S}_n]$*, where* $\mathbf{C}$ *is some (implicit) system which exports* $n$ *interfaces labeled* $\mathbf{S}_1, \ldots, \mathbf{S}_n$*.*[5]

### 2.3 Indistinguishability of Cryptographic Systems

In cryptography, we are usually interested in how similarly two systems $\mathbf{S}$ and $\mathbf{T}$ (with matching interfaces) behave. Intuitively, the more indistinguishable their behavior is, the closer $\mathbf{S}$ and $\mathbf{T}$ are. We can measure such closeness by means of the indistinguishability between systems $\mathbf{S}$ and $\mathbf{T}$ from the perspective of a distinguisher $\mathbf{D}$ which interacts with either of them, and outputs the bit denoted by $\mathbf{DV}$, for $\mathbf{V} \in \{\mathbf{S}, \mathbf{T}\}$, indicating its guess as to which system it is interacting with, where the understanding is that 0 indicates $\mathbf{S}$ and 1 indicates $\mathbf{T}$.

**Definition 2.** *For distinguisher* $\mathbf{D}$ *and systems* $\mathbf{S}$ *and* $\mathbf{T}$*,* $\mathbf{D}$*'s* advantage *in distinguishing between* $\mathbf{S}$ *and* $\mathbf{T}$ *is*

$$\Delta^{\mathbf{D}}(\mathbf{S}, \mathbf{T}) \doteq \Pr[\mathbf{DS} = 0] - \Pr[\mathbf{DT} = 0].$$

[4] We sometimes abuse notation, and make $\mathbf{C}$ in some sense more explicit; for instance, given a system $\mathbf{E}_k$ which performs encryption under a fixed key $k$ (which we assume must be provided as first input to the system), if we consider a random variable $K$ over the key-space, then we denote by $\langle \mathbf{E}_K, \mathbf{E}_K \rangle$ the correlated parallel composition corresponding to the system $\mathbf{K}[\mathbf{E}_k, \mathbf{E}_k]$, where $\mathbf{K}$ merely samples a key $k$ from $K$, and feeds it to both systems (thus correlating them), and then emulates two encryption oracles which use the same (random) key $K$.

[5] Note that correlated parallel composition is merely syntactic construct, and we only use this notation throughout our paper for easier (and nicer) statements.

Moreover, in cryptography security statements are often conditional, as is the case for the present work. This means that, given two systems $\mathbf{S}$ and $\mathbf{T}$, we do not give a concrete value for the distinguishing advantage depending on a distinguisher $\mathbf{D}$, but rather relate this quantity to the distinguishing advantage of *another* distinguisher $\mathbf{D}'$ for two different systems $\mathbf{S}'$ and $\mathbf{T}'$. Such a relation should entail that if $\mathbf{S}'$ and $\mathbf{T}'$ are close (which usually can be either in turn related to the distinction between two further systems, or just crystallized as an *hardness assumption*), then so are $\mathbf{S}$ and $\mathbf{T}$. Such a relation can be carried out by using the same distinguisher for the two different distinction problems, but more in general usually requires a *reduction* system $\mathbf{C}$ which translates $\mathbf{S}'$ and $\mathbf{T}'$ into two systems $\mathbf{CS}'$ and $\mathbf{CT}'$ that, towards $\mathbf{D}$, behave similarly to $\mathbf{S}$ and $\mathbf{T}$, respectively. Turned around, this also means that $\mathbf{C}$ translates the distinguisher $\mathbf{D}$ for $\mathbf{S}$ and $\mathbf{T}$ into the (similarly good) distinguisher $\mathbf{D}' = \mathbf{DC}$ for $\mathbf{S}'$ and $\mathbf{T}'$.[6] Therefore, if we assume that no (efficient) distinguisher can have a good advantage in distinguishing $\mathbf{S}'$ and $\mathbf{T}'$, then so does $\mathbf{D}'$, and in turn also $\mathbf{D}$ in distinguishing $\mathbf{S}$ and $\mathbf{T}$. By Definition 2 and Equation 1, this in particular implies

$$\Delta^{\mathbf{D}}(\mathbf{S},\mathbf{T}) = \Delta^{\mathbf{D}}(\mathbf{CS}',\mathbf{CT}') = \Delta^{\mathbf{DC}}(\mathbf{S}',\mathbf{T}') = \Delta^{\mathbf{D}'}(\mathbf{S}',\mathbf{T}'),$$

which we will extensively use in our proofs.

We next list some lemmas and definitions which are useful for proving the above mentioned relations. Given a distinguisher $\mathbf{D}$, we sometimes need to consider reduction systems which flip the bit output by $\mathbf{D}$. For this, we introduce a special converter $\mathbf{I}$.

**Definition 3.** *The* inversion converter $\mathbf{I}$ *is defined such that for any distinguisher* $\mathbf{D}$ *and any system* $\mathbf{S}$, $\mathbf{DIS} = 0 \iff \mathbf{DS} = 1$.

Then, the following trivial lemma follows easily by the above definition.

**Lemma 1.** *For distinguisher* $\mathbf{D}$ *and systems* $\mathbf{S}$ *and* $\mathbf{T}$, $\Delta^{\mathbf{D}}(\mathbf{S},\mathbf{T}) = \Delta^{\mathbf{DI}}(\mathbf{T},\mathbf{S})$.

The following two lemmas are usually used in conjunction when doing a so-called *hybrid argument*.

**Lemma 2.** *For distinguisher* $\mathbf{D}$ *and systems* $\mathbf{S}_1,\ldots,\mathbf{S}_n$,

$$\Delta^{\mathbf{D}}(\mathbf{S}_1,\mathbf{S}_n) = \sum_{i=1}^{n-1} \Delta^{\mathbf{D}}(\mathbf{S}_i,\mathbf{S}_{i+1}).$$

**Lemma 3.** *For distinguishers* $\mathbf{D}_1,\ldots,\mathbf{D}_n$, *systems* $\mathbf{S}$ *and* $\mathbf{T}$, *and random variable* $I$ *uniformly distributed over* $[n]$,

$$\sum_{i=1}^{n} \Delta^{\mathbf{D}_i}(\mathbf{S},\mathbf{T}) = n \cdot \Delta^{\mathbf{D}_I}(\mathbf{S},\mathbf{T}).$$

[6] In this work, we assume that such translations (reductions) are *black-box*, that is, $\mathbf{C}$ only has access to the outputs of $\mathbf{D}$, not to its internal behavior.

### 2.4 Probabilistic (Authenticated) Encryption (pE/pAE)

Syntactically, *probabilistic encryption* (pE) and *probabilistic authenticated encryption* (pAE) are the same object, which we generally call an *encryption scheme*. The distinction is merely on the level of security: if an encryption scheme provides *confidentiality* (or is IND-CPA-*secure*), we consider it *secure* pE, whereas if it provides *both confidentiality and authenticity* (or is IND-CCA3-*secure*), we consider it *secure* pAE.

**Definition 4 (Encryption Scheme).** *A* (probabilistic) encryption scheme $\Pi \doteq (\mathtt{Gen}, \mathtt{Enc}, \mathtt{Dec})$ *over key-space* $\mathcal{K}$*, message-space* $\mathcal{M}$*, and ciphertext-space* $\mathcal{C}$ *(with* $\perp \notin \mathcal{K} \cup \mathcal{M} \cup \mathcal{C}$*), is such that*

- $\mathtt{Gen}$ *is an (efficiently samplable) distribution over* $\mathcal{K}$*;*
- $\mathtt{Enc} : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$ *is a (efficiently computable) probabilistic function;*
- $\mathtt{Dec} : \mathcal{K} \times \mathcal{C} \to \mathcal{M} \cup \{\perp\}$ *is an (efficiently computable) deterministic function.*

*As customary, for* $k \in \mathcal{K}$ *we use the short-hand notation* $\mathtt{Enc}_k(\cdot)$ *for* $\mathtt{Enc}(k, \cdot)$ *and* $\mathtt{Dec}_k(\cdot)$ *for* $\mathtt{Dec}(k, \cdot)$*, and we also assume that* $\mathcal{M} \subseteq \{0,1\}^*$ *and for any* $m \in \mathcal{M}$*,* $\{0,1\}^{|m|} \subseteq \mathcal{M}$*, whereas* $\mathcal{C} = \{0,1\}^*$*, but for any* $m \in \mathcal{M}$ *and* $k \in \mathcal{K}$*,* $|\mathtt{Enc}_k(m)| = |m| + \tau$ *for some fixed* expansion factor $\tau \in \mathbb{N}$*. Moreover, we assume* correctness *of* $\Pi$*, that is, for all keys* $k$ *distributed according to* $\mathtt{Gen}$*, and all ciphertexts* $c \in \mathcal{C}$*,*

$$\mathtt{Dec}_k(c) = \begin{cases} m & \text{if } c \in \operatorname{supp}(\mathtt{Enc}_k(m)), \\ \perp & \text{otherwise.} \end{cases}$$

In order to define the security (and later also anonymity) of a fixed scheme $\Pi$, we define the following single and double interface systems (where the dependency on $\Pi$ is implicit), parameterized by a fixed key $k \in \mathcal{K}$:

- $\mathbf{E}_k$: On input a message $m \in \mathcal{M}$, return $\mathtt{Enc}_k(m) \in \mathcal{C}$.
- $\mathbf{E}_k^{\$}$: On input a message $m \in \mathcal{M}$, return $\mathtt{Enc}_k(\tilde{m}) \in \mathcal{C}$ for freshly and uniformly sampled $\tilde{m} \in \mathcal{M}$ with $|\tilde{m}| = |m|$.
- $\langle \mathbf{E}_k, \mathbf{D}_k \rangle$:
  - On input a message $m \in \mathcal{M}$, return $\mathtt{Enc}_k(m) \in \mathcal{C}$.
  - On input a ciphertext $c \in \mathcal{C}$, return $\mathtt{Dec}_k(c) \in \mathcal{M} \cup \{\perp\}$.
- $\langle \mathbf{E}_k, \mathbf{D}^{\perp} \rangle$: Initially set $\mathcal{Q} \subseteq \mathcal{M} \times \mathcal{C}$ to $\varnothing$ and then:
  - On input a message $m \in \mathcal{M}$, return $c \doteq \mathtt{Enc}_k(m) \in \mathcal{C}$ and set $\mathcal{Q}$ to $\mathcal{Q} \cup \{(m, c)\}$.
  - On input a ciphertext $c \in \mathcal{C}$, if there is an $m \in \mathcal{M}$ such that $(m, c) \in \mathcal{Q}$, then return $m$, otherwise return $\perp$.
- $\langle \mathbf{E}_k^{\$}, \mathbf{D}^{\perp} \rangle$: Initially set $\mathcal{Q} \subseteq \mathcal{M} \times \mathcal{C}$ to $\varnothing$ and then:
  - On input a message $m \in \mathcal{M}$, return $c \doteq \mathtt{Enc}_k(\tilde{m}) \in \mathcal{C}$ for freshly and uniformly sampled $\tilde{m} \in \mathcal{M}$ with $|\tilde{m}| = |m|$, and set $\mathcal{Q}$ to $\mathcal{Q} \cup \{(m, c)\}$.
  - On input a ciphertext $c \in \mathcal{C}$, if there is an $m \in \mathcal{M}$ such that $(m, c) \in \mathcal{Q}$, then return $m$, otherwise return $\perp$.

In our definitions, the key $k$ will *always* be replaced by a random variable (usually denoted $K$ or $K_i$, for some $i \in \mathbb{N}$) distributed according to $\Pi$'s `Gen`.

We remark that in our security definitions below we will slightly abuse notation and informally refer to *efficient* distinguishers and *negligible* advantages; both concepts should be properly defined asymptotically, which we do not explicitly do, since we do not define any *security parameter*. Nevertheless, correct asymptotic security statements may be easily recovered by considering sequences of our security statements, and taking the limit. Still, when relating such definitions, we will not (need to) use such asymptotic concepts, since we will employ a *concrete approach*, as done for example by Bellare, Desai, Jokipii, and Rogaway [BDJR97].

### 2.5 Game-Based Security of pE/pAE

Following [BDJR97], we first define the game-based security of pE in the *real-or-random* fashion, where the adversary must distinguish between a true encryption oracle and one which ignores inputs and encrypts random messages of the same length instead. For this reason we interchangeably talk about adversary and distinguisher. The following definition captures[7] well-known IND-CPA security notions commonly found in the literature.

**Definition 5 (Game-Based Security of pE).** *An encryption scheme* $\Pi$ *is* secure pE *(or* IND-CPA-secure*) if*

$$\Delta^{\mathbf{D}}(\mathbf{E}_K, \mathbf{E}_K^{\$})$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

For pAE we closely follow the *all-in-one* security definition style originally introduced by Shrimpton in [Shr04] and dubbed IND-CCA3, where an adversary must distinguish between two sets of oracles: the first set consists of true encryption and decryption oracles, whereas the second set consists of a fake encryption oracle which ignores inputs and encrypts random messages of the same length instead, and a fake decryption oracle which always return $\perp$, except if the provided ciphertext was previously output upon (fake) encryption, in which case the original message is returned. Note that this is actually a slightly different version than Shrimpton's original definition, and was put forth in [AGM18] by Alagic, Gagliardoni, and Majenz, where the equivalence with the former is shown.

**Definition 6 (Game-Based Security of pAE).** *An encryption scheme* $\Pi$ *is* secure pAE *(or* IND-CCA3-secure*) if*

$$\Delta^{\mathbf{D}}(\langle \mathbf{E}_K, \mathbf{D}_K \rangle, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp} \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

[7] It is actually equivalent to the one named ROR-CPA in [BDJR97], which is linearly equivalent (in the number of queries) to the one dubbed FTG-CPA (for "find-then-guess") therein, commonly referred to as IND-CPA in the literature.

## 3 Game-Based Anonymous Security of pE/pAE

We define game-based anonymity of pE and pAE in terms of what in the literature is usually termed *key-indistinguishability*. For this, recall from our discussion above (see Figure 1) that the system $[\mathbf{S}_{K_1}, \ldots, \mathbf{S}_{K_n}]$ provides the distinguisher with $n$ interfaces to $n$ *distinct* and *independent* copies of system $\mathbf{S}_k$, each of which is parameterized by a *different*, freshly and independently sampled key $K_i$. On the other hand, the system $\langle \mathbf{S}_K, \ldots, \mathbf{S}_K \rangle$ provides the distinguisher with $n$ interfaces to essentially the *same* copy of system $\mathbf{S}_k$, each of which is parameterized by the *same* key $K$ (previously freshly sampled).

We begin by providing a game-based security definition capturing exclusively the notion of anonymity (in terms of key-indistinguishability) of pE and pAE. In the following, when dropping the term [$n$-] we mean *"for any integer $n \geq 2$"*.

**Definition 7 (Game-Based Anonymity of pE).** *An encryption scheme* $\Pi$ *is* [$n$-]anonymous pE *(or* [$n$-]IK-CPA-secure*) if*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

**Definition 8 (Game-Based Anonymity of pAE).** *An encryption scheme* $\Pi$ *is* [$n$-]anonymous pAE *(or* [$n$-]IK-CCA3-secure*) if*

$$\Delta([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1} \rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n} \rangle], \langle \langle \mathbf{E}_K, \mathbf{D}^{\perp} \rangle, \ldots, \langle \mathbf{E}_K, \mathbf{D}^{\perp} \rangle \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

Next, we define the coupling of the traditional security goal of pE/pAE with anonymity. For both notions, we use the term *anonymous security*; specifically, by anonymous and secure pE we mean key-indistinguishable and confidential encryption, whereas by anonymous and secure pAE we mean key-indistinguishable, confidential, and authenticated encryption.

**Definition 9 (Game-Based Anonymous Security of pE).** *An encryption scheme* $\Pi$ *is* [$n$-]anonymous secure pE *(or* [$n$-]IND-IK-CPA-secure*) if*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

**Definition 10 (Game-Based Anonymous Security of pAE).** *An encryption scheme* $\Pi$ *is* [$n$-]anonymous secure pAE *(or* [$n$-]IND-IK-CCA3-secure*) if*

$$\Delta([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1} \rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n} \rangle], \langle \langle \mathbf{E}^{\$}_K, \mathbf{D}^{\perp} \rangle, \ldots, \langle \mathbf{E}^{\$}_K, \mathbf{D}^{\perp} \rangle \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

*Remarks.* The concept of key-indistinguishability has been first introduced under the name of *"key-hiding private-key encryption"* by Fischlin in [Fis99] as 2-IK-CPA according to Definition 7. Subsequently, in [Des00], Desai also studied the problem introducing the concept of *"non-separability of keys"*, but specifically for encryption schemes based on block ciphers. Later, in [AR00], Abadi and Rogaway presented a security notion called *"which-key concealing"*, that is basically identical to Fischlin's, but they defined security as a combination of key-indistinguishability and ciphertext-indistinguishability, that is, as 2-IND-IK-CPA according to Definition 9. They also claimed that popular modes of operation for symmetric encryption yield key-private encryption schemes. We will prove this formally in Subsection 3.2. Interestingly, the concept of key-indistinguishability was succesfully translated to the public-key setting by Bellare, Boldyreva, Desai, and Pointcheval in [BBDP01], where the terms *key-privacy* and *indistinguishability of keys* were originally suggested.

As previously mentioned, regarding key-indistinguishability of AE, in a very recent work Chan and Rogaway [CR19] introduce the nonce-based counterpart of our notion for pAE, Definition 10, which is crucially *not* directly applicable to nAE, but rather to anAE, a syntactically different scheme which can be obtained from nAE through the transformation NonceWrap that they introduce.

### 3.1 Relations Among Notions

In this section we show that the combination of ciphertext-indistinguishability (IND-{CPA,CCA3}) for pE/pAE and key-indistinguishability (IK-{CPA,CCA3}) for pE/pAE is equivalent to the respective game-based notion capturing both goals simultaneously (IND-IK-{CPA,CCA3}), regardless of the number of users. In order to keep the presentation simple, we drop the terms -CPA and -CCA3 from the textual description, and only formally show the relations for the case of pE: for any of the following results, the corresponding lifting to the case of pAE follows trivially.[8] Moreover, we defer all the proofs of this section to Appendix B.

We start by showing that key-indistinguishability is preserved up to constant increase when the number of users is incremented.

**Lemma 4.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle) = (n-1) \cdot \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K, \mathbf{E}_K \rangle).$$

*In particular, this implies that if an encryption scheme is* 2-IK-CPA*-secure, then it is also* $n$*-*IK-CPA*-secure.*

**Corollary 1.** *If an encryption scheme is* 2-IK-CCA3*-secure, then it is also* $n$*-*IK-CCA3*-secure.*

Next, we confirm the natural intuition that ciphertext-indistinguishability is also preserved when coupled with key-indistinguishability, that is, we show sufficiency of IND and IK security for IND-IK security.

[8] In the next sections we will however show two results that apply to both pE and pAE, thus illustrating how generally such lifting trivially follows.

**Lemma 5.** *For every distinguisher* $\mathbf{D}$*, there exist reductions* $\mathbf{C}$ *and* $\mathbf{C}'$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\ldots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K^{\$},\ldots,\mathbf{E}_K^{\$}\rangle) = (n-1)\cdot\Delta^{\mathbf{DC}}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle) + \Delta^{\mathbf{DC}'}(\mathbf{E}_K,\mathbf{E}_K^{\$}).$$

*In particular, this implies that if an encryption scheme is* 2-IK-CPA*-secure and* IND-CPA*-secure, then it is also* $n$-IND-IK-CPA*-secure.*

**Corollary 2.** *If an encryption scheme is* 2-IK-CCA3*-secure and* IND-CCA3*-secure, then it is also* $n$-IND-IK-CCA3*-secure.*

Note that similarly to Lemma 5, also $n$-IK security coupled with IND security implies $n$-IND-IK security. We now turn to the necessary conditions; first we show that indeed IND-IK security implies IND security.

**Lemma 6.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle) = 2\cdot\Delta^{\mathbf{DC}}([\mathbf{E}_{K_1},\ldots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K^{\$},\ldots,\mathbf{E}_K^{\$}\rangle).$$

*In particular, this implies that if an encryption scheme is* $n$-IND-IK-CPA*-secure, then it is also* 2-IK-CPA*-secure.*

**Corollary 3.** *If an encryption scheme is* $n$-IND-IK-CCA3*-secure, then it is also* 2-IK-CCA3*-secure.*

Note that similarly to Lemma 6, clearly $n$-IND-IK security also implies $n$-IK security. The last necessary condition is that IND-IK security implies IND security.

**Lemma 7.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}(\mathbf{E}_K,\mathbf{E}_K^{\$}) = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1},\ldots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K^{\$},\ldots,\mathbf{E}_K^{\$}\rangle).$$

*In particular, this implies that if an encryption scheme is* $n$-IND-IK-CPA*-secure, then it is also* IND-CPA*-secure.*

**Corollary 4.** *If an encryption scheme is* $n$-IND-IK-CCA3*-secure, then it is also* IND-CCA3*-secure.*

Therefore, we showed that an encryption scheme is ($n$-)IND-IK secure if and only if it is both ($n$-)IK and IND secure. Clearly this can be casted down to the 2 users case, in line with the security definitions of [Fis99,AR00].

**Corollary 5.** *An encryption scheme is* 2-IND-IK-CPA*-secure if and only if it is both* IND-CPA*-secure and* 2-IK-CPA*-secure.*

**Corollary 6.** *An encryption scheme is* 2-IND-IK-CCA3*-secure if and only if it is both* IND-CCA3*-secure and* 2-IK-CCA3*-secure.*

### 3.2 Computationally Uniform Ciphertexts Imply Anonymity

In this section we revisit a stronger security notion for symmetric encryption, which we call *indistinguishability from uniform ciphertexts*, *strong security*, or IND$-{CPA,CCA3}*-security*, and show a simple folklore result that was stated in [AR00] (of which, to the best of our knowledge, there is no formal proof yet). This definition intuitively should capture indistinguishability of ciphertexts, but it actually overshoots this goal, and it is stronger in the sense that it also implies indistinguishability of keys. Recall that IND-{CPA,CCA3}-security *does not* imply indistinguishability of keys, but it turns out to be easier to prove that schemes meet the stronger notion, which is also conceptually simpler. Essentially, instead of choosing a random message to be encrypted in the ideal world, a random ciphertext is output (thus neglecting encryption altogether).

This stronger security notion appears to have been originally introduced by Rogaway, Bellare, Black, and Krovetz in [RBBK01] for proving the security of the so-called offset codebook (OCB) mode of operation for symmetric encryption.[9] A number of other important results, such as the security of counter (CTR) or cipher block chaining (CBC) modes, first carried out in [BDJR97], have been later adapted by Rogaway [Rog04] to show that such schemes actually satisfy this stronger definition.[10] In fact, as argued in [AR00] (where this security notion—targeted to encryption rather than authenticated encryption—is dubbed *type-1 security*), by the above mentioned folklore result which we prove here, such modes indeed yield key indistinguishable schemes. We remark that subsequently, this definition was also used in the field of *provable secure steganography* (for both symmetric-key and asymmetric-key schemes) [HLvA02,vAH04,Möl04,BC05]. In the literature, this definition is alternatively called *indistinguishability from random bits/bitstrings* or simply *pseudorandom ciphertexts* security.

In order to formalize this notion, we need to introduce the system $\mathbf{\$}$ (with implicit dependency on a fixed encryption scheme $\Pi$) which on input any message $m \in \mathcal{M}$ simply outputs a uniformly sampled ciphertext of appropriate length, that is, according to our Definition 4, a uniform random bitstring of length $|m| + \tau$, where $\tau \in \mathbb{N}$ is the expansion factor defined by $\Pi$ (thus, in particular, $\mathbf{\$}$ does not make use of the underlying encryption function defined by $\Pi$). Then for the case of pE we can increase the security requirement as follows.

**Definition 11 (Game-Based Strong Security of pE).** *An encryption scheme* $\Pi$ *is* strongly secure pE *(or* IND$-CPA-secure*) if*

$$\Delta^{\mathbf{D}}(\mathbf{E}_K, \mathbf{\$})$$

*is negligible for any efficient distinguisher* $\mathbf{D}$.

[9] Note that OCB actually yields more than a secure encryption scheme: in [RBBK01] it is actually shown that OCB is *confidential* according to the mentioned stronger notion, but also *authentic*, thus making it a *secure authenticated encryption* scheme.

[10] All of those results are actually geared towards *nonce-based symmetric encryption*, but they also apply to our setting.

The analogous notion for pAE was introduced by Rogaway and Shrimpton in [RS06], and is adapted within our framework as follows.

**Definition 12 (Game-Based Strong Security of pAE).** *An encryption scheme* $\Pi$ *is* strongly secure pAE *(or* IND$-CCA3-secure*) if*

$$\Delta^{\mathbf{D}}(\langle \mathbf{E}_K, \mathbf{D}_K \rangle, \langle \$, \mathbf{D}^{\perp} \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

Next, starting with the case of pE, we show that the stronger notion of IND$-CPA indeed implies IND-IK-CPA (and thus also both IK-CPA and IND-CPA), as originally pointed out in [AR00]. This is captured formally by the following statement, shown for 2 users for cleaner presentation, but easily generalized to $n$ users.

**Theorem 1.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K^{\$}, \mathbf{E}_K^{\$} \rangle) = 3 \cdot \Delta^{\mathbf{DC}}(\mathbf{E}_K, \$).$$

*In particular, this implies that if an encryption scheme is* IND$-CPA*-secure, then it is also* IND-IK-CPA*-secure.*

*Proof.* For reduction systems $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$, such that, for any compatible system $\mathbf{S}$, $\mathbf{C}_1\mathbf{S} = [\mathbf{S}, \mathbf{E}_K]$, $\mathbf{C}_2\mathbf{S} = [\$, \mathbf{S}]$, and $\mathbf{C}_3\mathbf{S} = \langle \mathbf{S}^{\$}, \mathbf{S}^{\$} \rangle$, and for any distinguisher $\mathbf{D}$, by Lemma 2, Lemma 1, and Lemma 3,

$$\begin{aligned}
&\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K^{\$}, \mathbf{E}_K^{\$} \rangle) \\
&\quad = \Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], [\$, \mathbf{E}_K]) + \Delta^{\mathbf{D}}([\$, \mathbf{E}_K], [\$, \$]) + \Delta^{\mathbf{D}}([\$, \$], \langle \mathbf{E}_K^{\$}, \mathbf{E}_K^{\$} \rangle) \\
&\quad = \Delta^{\mathbf{D}}(\mathbf{C}_1\mathbf{E}_{K_1}, \mathbf{C}_1\$) + \Delta^{\mathbf{D}}(\mathbf{C}_2\mathbf{E}_K, \mathbf{C}_2\$) + \Delta^{\mathbf{D}}(\mathbf{C}_3\$, \mathbf{C}_3\mathbf{E}_K) \\
&\quad = \Delta^{\mathbf{DC}_1}(\mathbf{E}_K, \$) + \Delta^{\mathbf{DC}_2}(\mathbf{E}_K, \$) + \Delta^{\mathbf{DIC}_3}(\mathbf{E}_K, \$) \\
&\quad = 3 \cdot \Delta^{\mathbf{DC}'_I}(\mathbf{E}_K, \$),
\end{aligned}$$

where $\mathbf{C}'_1 \doteq \mathbf{C}_1$, $\mathbf{C}'_2 \doteq \mathbf{C}_2$, $\mathbf{C}'_3 \doteq \mathbf{IC}_3$, $I$ is uniformly distributed over $\{1, 2, 3\}$, and we used that $\mathbf{C}_3\$ = \langle \$, \$ \rangle \equiv [\$, \$]$. With $\mathbf{C} \doteq \mathbf{C}'_I$, this concludes the proof. □

Finally, the analogous statement for the case of pAE just follows as a natural lifting of Theorem 1, but since we consider this result more important than the previous relations among notions, instead of only providing a corollary we actually state the whole theorem with its proof, that is, we show that the stronger notion of IND$-CCA3 indeed implies IND-IK-CCA3 (and thus also both IK-CCA3 and IND-CCA3). We remark that this fact was informally pointed out by Rogaway [Rog13].

**Theorem 2.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\begin{aligned}&\Delta([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1}\rangle, \langle \mathbf{E}_{K_2}, \mathbf{D}_{K_2}\rangle], \langle\langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle\rangle)\\ &\quad = 3 \cdot \Delta^{\mathbf{DC}}(\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \langle \$, \mathbf{D}^{\perp}\rangle).\end{aligned}$$

*In particular, this implies that if an encryption scheme is* IND$-CCA3*-secure, then it is also* IND-IK-CCA3*-secure.*

*Proof.* For reduction systems $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$, such that, for any compatible system $\langle \mathbf{S}, \mathbf{T}\rangle$, $\mathbf{C}_1\langle \mathbf{S}, \mathbf{T}\rangle = [\langle \mathbf{S}, \mathbf{T}\rangle, \langle \mathbf{E}_K, \mathbf{D}_K\rangle]$, $\mathbf{C}_2\langle \mathbf{S}, \mathbf{T}\rangle = [\langle \$, \mathbf{D}^{\perp}\rangle, \langle \mathbf{S}, \mathbf{T}\rangle]$, and $\mathbf{C}_3\langle \mathbf{S}, \mathbf{T}\rangle = \langle\langle \mathbf{S}^{\$}, \mathbf{T}^{\perp}\rangle, \langle \mathbf{S}^{\$}, \mathbf{T}^{\perp}\rangle\rangle$, and for any distinguisher $\mathbf{D}$, by Lemma 2, Lemma 1, and Lemma 3,

$$\begin{aligned}&\Delta^{\mathbf{D}}([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1}\rangle, \langle \mathbf{E}_{K_2}, \mathbf{D}_{K_2}\rangle], \langle\langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle\rangle)\\ &\quad = \Delta^{\mathbf{D}}([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1}\rangle, \langle \mathbf{E}_{K_2}, \mathbf{D}_{K_2}\rangle], [\langle \$, \mathbf{D}^{\perp}\rangle, \langle \mathbf{E}_K, \mathbf{D}_K\rangle])\\ &\qquad + \Delta^{\mathbf{D}}([\langle \$, \mathbf{D}^{\perp}\rangle, \langle \mathbf{E}_K, \mathbf{D}_K\rangle], [\langle \$, \mathbf{D}^{\perp}\rangle, \langle \$, \mathbf{D}^{\perp}\rangle])\\ &\qquad + \Delta^{\mathbf{D}}([\langle \$, \mathbf{D}^{\perp}\rangle, \langle \$, \mathbf{D}^{\perp}\rangle], \langle\langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp}\rangle\rangle)\\ &\quad = \Delta^{\mathbf{D}}(\mathbf{C}_1\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \mathbf{C}_1\langle \$, \mathbf{D}^{\perp}\rangle) + \Delta^{\mathbf{D}}(\mathbf{C}_2\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \mathbf{C}_2\langle \$, \mathbf{D}^{\perp}\rangle)\\ &\qquad + \Delta^{\mathbf{D}}(\mathbf{C}_3\langle \$, \mathbf{D}^{\perp}\rangle, \mathbf{C}_3\langle \mathbf{E}_K, \mathbf{D}_K\rangle)\\ &\quad = \Delta^{\mathbf{DC}_1}(\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \langle \$, \mathbf{D}^{\perp}\rangle) + \Delta^{\mathbf{DC}_2}(\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \langle \$, \mathbf{D}^{\perp}\rangle)\\ &\qquad + \Delta^{\mathbf{DIC}_3}(\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \langle \$, \mathbf{D}^{\perp}\rangle)\\ &\quad = 3 \cdot \Delta^{\mathbf{DC}'_I}(\langle \mathbf{E}_K, \mathbf{D}_K\rangle, \langle \$, \mathbf{D}^{\perp}\rangle),\end{aligned}$$

where $\mathbf{C}'_1 \doteq \mathbf{C}_1$, $\mathbf{C}'_2 \doteq \mathbf{C}_2$, $\mathbf{C}'_3 \doteq \mathbf{IC}_3$, $I$ is uniformly distributed over $\{1, 2, 3\}$, and we used that $\mathbf{C}_3\langle \$, \mathbf{D}^{\perp}\rangle = \langle\langle \$, \mathbf{D}^{\perp}\rangle, \langle \$, \mathbf{D}^{\perp}\rangle\rangle \equiv [\langle \$, \mathbf{D}^{\perp}\rangle, \langle \$, \mathbf{D}^{\perp}\rangle]$. With $\mathbf{C} \doteq \mathbf{C}'_I$, this concludes the proof. □

### 3.3 Anonymity Preservation of Encrypt-then-MAC

After having related the various game-based notions for pE and for pAE separately, we finally show how the anonymity enhanced security definitions for pE relate with those of pAE. For this, we need to introduce the concept of *message authentication code (MAC)* and its security and anonymity notions, which we defer to Appendix C. Recall that Bellare and Namprempre [BN00] and Krawczyk [Kra01] have shown that the combination of an unforgeable (UF-CMA) MAC and a secure (IND-CPA) encryption scheme, performed according to the *Encrypt-then-MAC* (EtM) paradigm, yields an encryption scheme which is both secure (IND-CPA) and unforgeable (INT-CTXT, the equivalent notion of UF-CMA for encryption). Later, Shrimpton [Shr04] showed that a nice *all-in-one* security definition for secure authenticated encryption, IND-CCA3, is equivalent to the combination IND-CPA and INT-CTXT, thus attesting that EtM performed on a UF-CMA-secure MAC scheme and an IND-CPA-secure encryption scheme, yields a IND-CCA3-secure authenticated encryption scheme. Using our notation from Subsection 2.4 and

Appendix C, the encryption scheme $\mathtt{EtM}(\Pi, \Sigma) \doteq (\widehat{\mathtt{Gen}}, \widehat{\mathtt{Tag}}, \widehat{\mathtt{Vrf}})$, resulting from this specific composition of an encryption scheme $\Pi \doteq (\mathtt{Gen}_\Pi, \mathtt{Enc}, \mathtt{Dec})$ (with key-space $\mathcal{K}_\Pi$) and a MAC scheme $\Sigma \doteq (\mathtt{Gen}_\Sigma, \mathtt{Tag}, \mathtt{Vrf})$ (with key-space $\mathcal{K}_\Sigma$) is defined as follows:[11]

- $\widehat{\mathtt{Gen}}$ is the product distribution of $\mathtt{Gen}_\Pi$ and $\mathtt{Gen}_\Sigma$ over $\mathcal{K}_\Pi \times \mathcal{K}_\Sigma$;
- $\widehat{\mathtt{Enc}}_{(k_\mathrm{e},k_\mathrm{a})} \doteq \mathtt{Tag}_{k_\mathrm{a}} \circ \mathtt{Enc}_{k_\mathrm{e}}$;
- $\widehat{\mathtt{Vrf}}_{(k_\mathrm{e},k_\mathrm{a})} \doteq \mathtt{Dec}_{k_\mathrm{e}} \circ \mathtt{Vrf}_{k_\mathrm{a}}$.

Note that in order for correctness to hold, we further need to assume that $\perp \in \mathcal{M}$, and that $\mathtt{Enc}_k(\perp) = \perp$ for any $k \in \mathcal{K}_\Pi$.

If we now want to define security of the composed scheme $\widehat{\Pi} \doteq \mathtt{EtM}(\Pi, \Sigma)$, we need to introduce a simple operator between (single-interface) systems, namely *cascading*: Informally, given systems $\mathbf{S}$ and $\mathbf{T}$, we define the new system $\mathbf{S}\triangleright\mathbf{T}$ as the system that on input $x$ computes $y \doteq \mathbf{S}(x)$, and returns $z \doteq \mathbf{T}(y)$ (where we are assuming matching domains). As we did for $\mathsf{pE}$, we can define systems $\mathbf{T}_k$ and $\mathbf{V}_k$ relative to MAC scheme $\Sigma$. Then $\widehat{\mathtt{Enc}}_{(k_\mathrm{e},k_\mathrm{a})}$ is modeled by $\widehat{\mathbf{E}}_{(k_\mathrm{e},k_\mathrm{a})} \doteq \mathbf{E}_{k_\mathrm{e}}\triangleright\mathbf{T}_{k_\mathrm{a}}$, and $\widehat{\mathtt{Dec}}_{(k_\mathrm{e},k_\mathrm{a})}$ by $\widehat{\mathbf{D}}_{(k_\mathrm{e},k_\mathrm{a})} \doteq \mathbf{V}_{k_\mathrm{a}}\triangleright\mathbf{D}_{k_\mathrm{e}}$. Recalling the security definitions from Subsection 2.4 and Appendix C, the statement that $\widehat{\Pi}$ is secure follows.

**Theorem 3.** *For every distinguisher $\mathbf{D}$, there exist reductions $\mathbf{C}$ and $\mathbf{C}'$ such that*

$$\Delta^{\mathbf{D}}(\langle\widehat{\mathbf{E}}_K, \widehat{\mathbf{D}}_K\rangle, \langle\widehat{\mathbf{E}}^{\$}_K, \widehat{\mathbf{D}}^{\perp}\rangle) = \Delta^{\mathbf{DC}}(\mathbf{E}_K, \mathbf{E}^{\$}_K) + \Delta^{\mathbf{DC}'}(\langle\mathbf{T}_K, \mathbf{V}_K\rangle, \langle\mathbf{T}_K, \mathbf{V}^{\perp}\rangle).$$

*In particular, this implies that if $\Pi$ is* IND-CPA*-secure and $\Sigma$ is* UF-CMA*-secure, then $\mathtt{EtM}(\Pi, \Sigma)$ is* IND-CCA3*-secure.*

*Proof.* For reduction system $\mathbf{C}$ such that, for any compatible system $\mathbf{E}$, $\mathbf{CE} = \langle\mathbf{E}\triangleright\mathbf{T}_K, \widehat{\mathbf{D}}^{\perp}\rangle$, and reduction system $\mathbf{C}'$ such that, for any compatible system $\langle\mathbf{T}, \mathbf{V}\rangle$, $\mathbf{C}'\langle\mathbf{T}, \mathbf{V}\rangle = \langle\mathbf{E}_K\triangleright\mathbf{T}, \mathbf{V}\triangleright\mathbf{D}_K\rangle$, and for any distinguisher $\mathbf{D}$, by Lemma 2,

$$\begin{aligned}
\Delta^{\mathbf{D}}(\langle\widehat{\mathbf{E}}_K, \widehat{\mathbf{D}}_K\rangle, \langle\widehat{\mathbf{E}}^{\$}_K, \widehat{\mathbf{D}}^{\perp}\rangle) &= \Delta^{\mathbf{D}}(\langle\widehat{\mathbf{E}}_K, \widehat{\mathbf{D}}_K\rangle, \langle\widehat{\mathbf{E}}_K, \widehat{\mathbf{D}}^{\perp}\rangle) \\
&\quad + \Delta^{\mathbf{D}}(\langle\widehat{\mathbf{E}}_K, \widehat{\mathbf{D}}^{\perp}\rangle, \langle\widehat{\mathbf{E}}^{\$}_K, \widehat{\mathbf{D}}^{\perp}\rangle) \\
&= \Delta^{\mathbf{D}}(\mathbf{C}'\langle\mathbf{T}_K, \mathbf{V}_K\rangle, \mathbf{C}'\langle\mathbf{T}_K, \mathbf{V}^{\perp}\rangle) \\
&\quad + \Delta^{\mathbf{D}}(\mathbf{CE}_K, \mathbf{CE}^{\$}_K) \\
&= \Delta^{\mathbf{DC}}(\mathbf{E}_K, \mathbf{E}^{\$}_K) + \Delta^{\mathbf{DC}'}(\langle\mathbf{T}_K, \mathbf{V}_K\rangle, \langle\mathbf{T}_K, \mathbf{V}^{\perp}\rangle).
\end{aligned}$$

This concludes the proof. □

We finally show the important fact that EtM is *anonymity-preserving*, in the sense that if an encryption scheme $\Pi$ is both IND-CPA-secure and IK-CPA-secure (that is, IND-IK-CPA-secure) and a MAC scheme $\Sigma$ is both UF-CMA-secure and IK-CMA-secure (that is, UF-IK-CMA-secure), then $\mathtt{EtM}(\Pi, \Sigma)$ not only is

[11] Recall that the symbol $\circ$ in this context represents *function composition*.

IND-CCA3-secure, but also IK-CCA3-secure (that is, IND-IK-CCA3-secure). This is captured formally by the following statement, shown for 2 users for cleaner presentation, but easily generalized to $n$ users.

**Theorem 4.** *For every distinguisher* $\mathbf{D}$*, there exist reduction* $\mathbf{C}$ *and* $\mathbf{C}'$ *such that*

$$\begin{aligned}
&\Delta^{\mathbf{D}}([\langle \widehat{\mathbf{E}}_{K_1}, \widehat{\mathbf{D}}_{K_1}\rangle, \langle \widehat{\mathbf{E}}_{K_2}, \widehat{\mathbf{D}}_{K_2}\rangle], \langle\langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle\rangle)\\
&\quad = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}^{\$}_{K}, \mathbf{E}^{\$}_{K}\rangle)\\
&\qquad + \Delta^{\mathbf{DC}'}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2}\rangle], \langle\langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle\rangle).
\end{aligned}$$

*In particular, this implies that if* $\Pi$ *is* IND-IK-CPA*-secure and* $\Sigma$ *is* UF-IK-CMA*-secure, then* $\mathtt{EtM}(\Pi, \Sigma)$ *is* IND-IK-CCA3*-secure.*

*Proof.* For reduction system $\mathbf{C}$ such that, for any compatible system $\langle \mathbf{E}_1, \mathbf{E}_2\rangle$, $\mathbf{C}\langle \mathbf{E}_1, \mathbf{E}_2\rangle = \langle\langle \mathbf{E}_1 \triangleright \mathbf{T}_K, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \mathbf{E}_2 \triangleright \mathbf{T}_K, \widehat{\mathbf{D}}^{\perp}\rangle\rangle$, and reduction system $\mathbf{C}'$ such that, for any compatible system $\langle\langle \mathbf{T}_1, \mathbf{V}_1\rangle, \langle \mathbf{T}_2, \mathbf{V}_2\rangle\rangle$, $\mathbf{C}'\langle\langle \mathbf{T}_1, \mathbf{V}_1\rangle, \langle \mathbf{T}_2, \mathbf{V}_2\rangle\rangle = \langle\langle \mathbf{E}_{K_1} \triangleright \mathbf{T}_1, \mathbf{V}_1 \triangleright \mathbf{D}_{K_1}\rangle, \langle \mathbf{E}_{K_2} \triangleright \mathbf{T}_2, \mathbf{V}_2 \triangleright \mathbf{D}_{K_2}\rangle\rangle$, and for any distinguisher $\mathbf{D}$, by Lemma 2,

$$\begin{aligned}
&\Delta^{\mathbf{D}}([\langle \widehat{\mathbf{E}}_{K_1}, \widehat{\mathbf{D}}_{K_1}\rangle, \langle \widehat{\mathbf{E}}_{K_2}, \widehat{\mathbf{D}}_{K_2}\rangle], \langle\langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle\rangle)\\
&\quad = \Delta^{\mathbf{D}}([\langle \widehat{\mathbf{E}}_{K_1}, \widehat{\mathbf{D}}_{K_1}\rangle, \langle \widehat{\mathbf{E}}_{K_2}, \widehat{\mathbf{D}}_{K_2}\rangle], [\langle \widehat{\mathbf{E}}_{K_1}, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \widehat{\mathbf{E}}_{K_2}, \widehat{\mathbf{D}}^{\perp}\rangle])\\
&\qquad + \Delta^{\mathbf{D}}([\langle \widehat{\mathbf{E}}_{K_1}, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \widehat{\mathbf{E}}_{K_2}, \widehat{\mathbf{D}}^{\perp}\rangle], \langle\langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle, \langle \widehat{\mathbf{E}}^{\$}_{K}, \widehat{\mathbf{D}}^{\perp}\rangle\rangle)\\
&\quad = \Delta^{\mathbf{D}}(\mathbf{C}'[\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2}\rangle], \mathbf{C}'\langle\langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle\rangle)\\
&\qquad + \Delta^{\mathbf{D}}(\mathbf{C}[\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \mathbf{C}\langle \mathbf{E}^{\$}_{K}, \mathbf{E}^{\$}_{K}\rangle)\\
&\quad = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}^{\$}_{K}, \mathbf{E}^{\$}_{K}\rangle)\\
&\qquad + \Delta^{\mathbf{DC}'}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2}\rangle], \langle\langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_{K}, \mathbf{V}^{\perp}\rangle\rangle).
\end{aligned}$$

This concludes the proof. □

We will confirm Theorem 4 with a composable approach in the next section.

## 4 Composable Security of Anonymous Communication

In this section we turn our attention to *composable security*, as opposed to game-based security. For this, we make use of the *Constructive Cryptography* (CC) framework by Maurer [Mau12], which is a specialization of the *Abstract Cryptography* theory by Maurer and Renner [MR11].

### 4.1 Constructive Cryptography

In essence, CC allows to define security of cryptographic protocols as statements about constructions of resources from other resources, which we model as cryptographic systems from Subsection 2.2. For such systems, we might at times use

suggestive words typed in sans-serif rather than bold-faced letters. The various interfaces of a resource should be thought of as being assigned to parties. In this work, all resources are parameterized by an integer $n \geq 2$ (the case $n = 1$ would be pointless for anonymity), and each defines $n + 2$ interfaces: $n$ for the *senders*, denoted $S_i$, for $i \in [n]$, one for the *adversary*, denoted $E$, and one for the *receiver*, denoted $R$. Therefore, in the following we use the expression $n$-resource to make explicit such parameter. Another crucial ingredient of CC are *converters*, also formally modeled as systems (labeled by lower-case sans-serif suggestive words), which when applied to interfaces of $n$-resources, give raise to a new $n$-resource. Within our formalization of cryptographic systems, CC converters thus correspond to converters of systems as defined in Subsection 2.2, but where we extend the sequential composition notion by allowing a (single-interface) converter system to be attached to just one of the interfaces of another $n$-resource system. Given a converter $\mathsf{cnv}$ and an $n$-resource $\mathbf{R}$, for $i \in [n]$ we denote the new $n$-resource system resulting from *attaching converter* $\mathsf{cnv}$ *to interface* $S_i$ *of* $n$*-resource* $\mathbf{R}$ as $\mathsf{cnv}^{S_i}\,\mathbf{R}$. Note that this automatically implies commutativity of converters attached to different interfaces, that is, considering a second converter $\widehat{\mathsf{cnv}}$ and letting $j \in [n]$ such that $j \neq i$, then $\mathsf{cnv}^{S_i}\,\widehat{\mathsf{cnv}}^{S_j}\,\mathbf{R} \equiv \widehat{\mathsf{cnv}}^{S_j}\,\mathsf{cnv}^{S_i}\,\mathbf{R}$.

In order to make security statements within CC, we model protocols as lists of converters. For $n$-resources, this means that a protocol $\pi$ executed by $n$ senders and one receiver (an $n$-protocol) is a list of $n + 1$ converters $(\mathsf{cnv}_1, \ldots, \mathsf{cnv}_{n+1})$, where the adopted convention is that $\mathsf{cnv}_i$ is attached to sender interface $S_i$, for $i \in [n]$, while $\mathsf{cnv}_{n+1}$ is attached to the receiver interface $R$. In the following, we use the short-hand notation $\pi\mathbf{R}$ for the $n$-resource $\mathsf{cnv}_1^{S_1} \cdots \mathsf{cnv}_n^{S_n}\,\mathsf{cnv}_{n+1}^{R}\,\mathbf{R}$. Moreover, for a second $n$-protocol $\hat{\pi} \doteq (\widehat{\mathsf{cnv}}_1, \ldots, \widehat{\mathsf{cnv}}_{n+1})$, we define the *composition* of $\hat{\pi}$ and $\pi$ as $\hat{\pi}\pi \doteq (\widehat{\mathsf{cnv}}_1\mathsf{cnv}_1, \ldots, \widehat{\mathsf{cnv}}_{n+1}\mathsf{cnv}_{n+1})$, and therefore $\hat{\pi}\pi\mathbf{R}$ is the $n$-resource $(\widehat{\mathsf{cnv}}_1\mathsf{cnv}_1)^{S_1} \cdots (\widehat{\mathsf{cnv}}_n\mathsf{cnv}_n)^{S_n}\,(\widehat{\mathsf{cnv}}_{n+1}\mathsf{cnv}_{n+1})^{R}\,\mathbf{R}$. The last ingredient we need is that of a simulator, which can be simply understood as a converter to be attached to the adversarial interface $E$. With this, we can now express composable security of an $n$-protocol $\pi$ in terms of indistinguishability as follows.

**Definition 13 (Construction).** *For* $n$*-resources* $\mathbf{R}$ *and* $\mathbf{S}$*, and function* $\varepsilon$ *mapping distinguishers to real values, we say that an* $n$*-protocol* $\pi$ constructs $\mathbf{S}$ from $\mathbf{R}$ within $\varepsilon$*, denoted*

$$\mathbf{R} \xrightarrow{\pi,\varepsilon} \mathbf{S},$$

*if there exists a simulator* $\mathsf{sim}$ *such that for all distinguishers* $\mathbf{D}$*,*

$$\Delta^{\mathbf{D}}(\pi\mathbf{R}, \mathsf{sim}^E\,\mathbf{S}) \leq \varepsilon(\mathbf{D}).$$

The intuition is that, if lifted to the asymptotic setting, Definition 13 implies that if $\varepsilon(\mathbf{D})$ is negligible for every efficient distinguisher $\mathbf{D}$, then the real $n$-resource $\mathbf{R}$ looks indistinguishable from the ideal $n$-resource $\mathbf{S}$. This naturally hints to the intuition that in any context where $\mathbf{S}$ is needed, $\pi\mathbf{R}$ can be safely used instead. This is the central point of composable security definitions, and is formalized by the following theorem, following directly from [MR11] (we nevertheless provide a short proof of this special for case here).

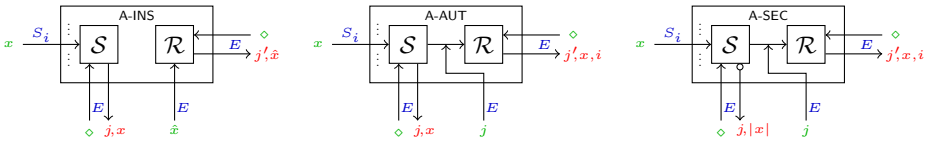


**Fig. 2:** Sketches of the anonymous channel resources (blue: interfaces; green: inputs; red: outputs).

**Theorem 5 (Composition).** *Let* $\mathbf{R}, \mathbf{S}, \mathbf{T}$ *be* $n$*-resources, and* $\pi_1, \pi_2$ $n$*-protocols. If* $\mathbf{R} \xrightarrow{\pi_1,\varepsilon_1} \mathbf{S}$ *and* $\mathbf{S} \xrightarrow{\pi_2,\varepsilon_2} \mathbf{T}$*, then* $\mathbf{R} \xrightarrow{\pi_2\pi_1,\hat{\varepsilon}_1\oplus\hat{\varepsilon}_2} \mathbf{T}$*, where* $\hat{\varepsilon}_1(\mathbf{D}) \doteq \varepsilon_1(\mathbf{D}\pi_2)$*,* $\hat{\varepsilon}_2(\mathbf{D}) \doteq \varepsilon_2(\mathbf{D}\,\mathsf{sim}_2^E)$*,* $\mathsf{sim}_2$ *is any simulator whose existence justifies* $\mathbf{S} \xrightarrow{\pi_2,\varepsilon_2} \mathbf{T}$*, and* $(\hat{\varepsilon}_1 \oplus \hat{\varepsilon}_2)(\mathbf{D}) \doteq \hat{\varepsilon}_1(\mathbf{D}) + \hat{\varepsilon}_2(\mathbf{D})$*.*

*Proof.* Recall that $\mathbf{R} \xrightarrow{\pi_1,\varepsilon_1} \mathbf{S}$ means that there exists a simulator $\mathsf{sim}_1$ such that for all distinguishers $\mathbf{D}$, $\Delta^{\mathbf{D}}(\pi_1\mathbf{R}, \mathsf{sim}_1^E\,\mathbf{S}) \leq \varepsilon_1(\mathbf{D})$, and $\mathbf{S} \xrightarrow{\pi_2,\varepsilon_2} \mathbf{T}$ means that there exists a simulator $\mathsf{sim}_2$ such that for all distinguishers $\mathbf{D}$, $\Delta^{\mathbf{D}}(\pi_2\mathbf{S}, \mathsf{sim}_2^E\,\mathbf{T}) \leq \varepsilon_2(\mathbf{D})$. Then, using commutativity of converters (attached to different interfaces) and Lemma 2, the theorem follows immediately by observing that

$$\begin{aligned}
\Delta^{\mathbf{D}}(\pi_2\pi_1\mathbf{R}, (\mathsf{sim}_1\mathsf{sim}_2)^E\,\mathbf{T}) &= \Delta^{\mathbf{D}}(\pi_2\pi_1\mathbf{R}, \pi_2\,\mathsf{sim}_1^E\,\mathbf{S}) \\
&\quad + \Delta^{\mathbf{D}}(\mathsf{sim}_1^E\,\pi_2\,\mathbf{S}, \mathsf{sim}_1^E\,\mathsf{sim}_2^E\,\mathbf{T}) \\
&= \Delta^{\mathbf{D}\pi_2}(\pi_1\mathbf{R}, \mathsf{sim}_1^E\,\mathbf{S}) + \Delta^{\mathbf{D}\,\mathsf{sim}_1^E}(\pi_2\,\mathbf{S}, \mathsf{sim}_2^E\,\mathbf{T}) \\
&\leq \varepsilon_1(\mathbf{D}\pi_2) + \varepsilon_2(\mathbf{D}\,\mathsf{sim}_1^E) \\
&= \hat{\varepsilon}_1(\mathbf{D}) + \hat{\varepsilon}_2(\mathbf{D}) \\
&= (\hat{\varepsilon}_1 \oplus \hat{\varepsilon}_2)(\mathbf{D}),
\end{aligned}$$

which by definition implies $\mathbf{R} \xrightarrow{\pi_2\pi_1,\hat{\varepsilon}_1\oplus\hat{\varepsilon}_2} \mathbf{T}$. □

**Anonymous Channels.** There are four $n$-resources that we consider in this work. The first, $\mathsf{KEY}_{\mathcal{K}}^n$, models the initial symmetric-key setup: it generates $n$ independent keys $K_1, \ldots, K_n \in \mathcal{K}$ according to an implicitly defined distribution $\mathtt{Gen}$ over $\mathcal{K}$, and for $i \in [n]$ it outputs $K_i$ at interface $S_i$; at interface $R$ it outputs the list $(K_1, \ldots, K_n)$ of all generated keys, while it outputs nothing at interface $E$. The remaining three $n$-resources model the anonymous channels for $n$ senders and one receiver mentioned above (for messages over some set $\mathcal{X}$), where we assume a central adversary that is in full control of the physical communication between the senders and the receiver, that is, an adversary that can *delete*, *repeat*, and *reorder* messages.[12] $\mathsf{A\text{-}INS}_{\mathcal{X}}^n$ models the channel which leaks every message

[12] Note that while deletion is a physical phenomenon, and can thus not be prevented using cryptography, it is in principle possible to prevent repetition and reordering, concretely by means of *sequence numbers*. But we do not cover this aspect of security in this work.

$\mathsf{A\text{-}INS}^n_{\mathcal{X}}$

$\mathcal{S}, \mathcal{R} \subseteq \mathbb{N} \times \mathcal{X}$,
$c_S, c_R, t_S, t_R \in \mathbb{N}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing$
  $c_S, c_R \leftarrow 1$
  $t_S, t_R \leftarrow 0$
**Interface** $S_i(x \in \mathcal{X})$:
  $t_S \leftarrow t_S + 1$
  $\mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, x)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, x) \in \mathcal{S} \mid c_S \leq j \leq t_S\}$
  $c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(x \in \mathcal{X})$:
  $t_R \leftarrow t_R + 1$
  $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, x)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, x) \in \mathcal{R} \mid c_R \leq j \leq t_R\}$
  $c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

$\mathsf{A\text{-}AUT}^n_{\mathcal{X}}$

$\mathcal{S}, \mathcal{R} \subseteq (\mathbb{N} \times \mathcal{X} \times \mathbb{N}) \cup (\mathbb{N} \times \{\bot\}^2)$,
$c_S, c_R, t_S, t_R \in \mathbb{N}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing, c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0$
**Interface** $S_i(x \in \mathcal{X})$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, x, i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, x) \in \mathbb{N} \times \mathcal{X} \mid$
    $\exists i \in [n] : (j, x, i) \in \mathcal{S},$
    $c_S \leq j \leq t_S\}$
  $c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(j \in \mathbb{N} \cup \{-1\})$:
  **if** $\exists x \in \mathcal{X}, i \in [n] : (j, x, i) \in \mathcal{S}$ **then**
    $t_R \leftarrow t_R + 1$
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, x, i)\}$
  **else if** $j = -1$ **then**
    $t_R \leftarrow t_R + 1$
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, \bot, \bot)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, x, i) \in \mathcal{R} \mid c_R \leq j \leq t_R\}$
  $c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

$\mathsf{A\text{-}SEC}^n_{\mathcal{X}}$

$\mathcal{S}, \mathcal{R} \subseteq (\mathbb{N} \times \mathcal{X} \times \mathbb{N}) \cup (\mathbb{N} \times \{\bot\}^2),\ c_S, c_R, t_S, t_R \in \mathbb{N}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0$
**Interface** $S_i(x \in \mathcal{X})$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, x, i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, |x|) \in \mathbb{N} \times \mathbb{N} \mid \exists i \in [n] : (j, x, i) \in \mathcal{S},\ c_S \leq j \leq t_S\},\ c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(j \in \mathbb{N} \cup \{-1\})$:
  **if** $\exists x \in \mathcal{X}, i \in [n] : (j, x, i) \in \mathcal{S}$ **then**
    $t_R \leftarrow t_R + 1,\ \mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, x, i)\}$
  **else if** $j = -1$ **then**
    $t_R \leftarrow t_R + 1,\ \mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, \bot, \bot)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, x, i) \in \mathcal{R} \mid c_R \leq j \leq t_R\},\ c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

**Fig. 3:** Formal description of the *insecure* ($\mathsf{A\text{-}INS}^n_{\mathcal{X}}$), *authenticated* ($\mathsf{A\text{-}AUT}^n_{\mathcal{X}}$), and *secure* ($\mathsf{A\text{-}SEC}^n_{\mathcal{X}}$) *anonymous channels*, each with the differences from the weaker one highlighted in blue.

input by any sender (but not their identities) directly to the adversary. Note that in particular this means that the receiver does not directly receive the messages sent by the senders. Moreover, $\mathsf{A\text{-}INS}^n_{\mathcal{X}}$ allows the adversary to inject any message to the receiver (thus, in particular, also the ones originally sent by the senders). Note that this channel, while providing anonymity, is per se pretty useless, since the receiver has also no information about the identity of the sender of any message. Instead, $\mathsf{A\text{-}AUT}^n_{\mathcal{X}}$, while still leaking all the messages sent by the senders directly to the adversary, does not allow the latter to inject any message; instead, the adversary can now *select* messages that it wants to be forwarded to the receiver. Moreover, the forwarded messages also carry the identity of the original sender, still hidden to the adversary. Finally, $\mathsf{A\text{-}SEC}^n_{\mathcal{X}}$ essentially works as $\mathsf{A\text{-}AUT}^n_{\mathcal{X}}$, except that now only the *lengths* of the messages sent by the senders are leaked directly to the adversary. We sketch the three anonymous channels in Figure 2 and provide a formal description of the behavior of the systems implementing such $n$-resources in Figure 3.

### 4.2 Overview of the Results

In [AHM+15] it was already shown[13] that UF-IK-CMA-secure pMAC constructs[14] A-AUT from A-INS and KEY; in Appendix C we restate the result within our model, which is captured by the following statement (cf. Theorem 8 therein):

$$[\mathsf{KEY}, \mathsf{A\text{-}INS}] \xrightarrow{\pi_{\mathsf{mac}}, \varepsilon_{\mathsf{mac}}} \mathsf{A\text{-}AUT},$$

(for appropriate $n$-protocol $\pi_{\mathsf{mac}}$ implementing pMAC and function $\varepsilon_{\mathsf{mac}}$). Here we instead focus on the following further constructions:

- IND-IK-CPA-secure pE constructs A-SEC from A-AUT and KEY (cf. Theorem 6):
$$[\mathsf{KEY}, \mathsf{A\text{-}AUT}] \xrightarrow{\pi_{\mathsf{enc}}, \varepsilon_{\mathsf{enc}}} \mathsf{A\text{-}SEC},$$
(for appropriate $n$-protocol $\pi_{\mathsf{enc}}$ and function $\varepsilon_{\mathsf{enc}}$).
- IND-IK-CCA3-secure pAE constructs A-SEC from A-INS and KEY (cf. Theorem 7):
$$[\mathsf{KEY}, \mathsf{A\text{-}INS}] \xrightarrow{\pi_{\mathsf{ae}}, \varepsilon_{\mathsf{ae}}} \mathsf{A\text{-}SEC},$$
(for appropriate $n$-protocol $\pi_{\mathsf{ae}}$ and function $\varepsilon_{\mathsf{ae}}$).

Note that by the composition theorem (Theorem 5), the first two statements imply the third for the (composed) protocol $\pi_{\mathsf{ae}} = \pi_{\mathsf{enc}}\pi_{\mathsf{mac}}$ and function $\varepsilon_{\mathsf{ae}} = \hat{\varepsilon}_{\mathsf{enc}} \oplus \hat{\varepsilon}_{\mathsf{mac}}$, namely

$$[\mathsf{KEY}, \mathsf{KEY}, \mathsf{A\text{-}INS}] \xrightarrow{\pi_{\mathsf{enc}}\pi_{\mathsf{mac}}, \hat{\varepsilon}_{\mathsf{enc}} \oplus \hat{\varepsilon}_{\mathsf{mac}}} \mathsf{A\text{-}SEC}.$$

In particular, note that this corresponds to the EtM paradigm, and therefore is a (composable) confirmation of Theorem 4.

[13] For a slightly different modeling of the anonymous channel resources.

[14] For better readability, in the following highlights of the results we drop the parameters of the involved channels; nevertheless, in the referenced formal results (which follow these highlights) we will make such parameters explicit.

### 4.3 Composable Anonymous Security of pE

In this section we first introduce a composable definition of anonymous security for pE, and then we show that the previously introduced game-based notion of IND-IK-CPA-security implies the former. The composable definition can be interpreted as providing *composable semantics* to IND-IK-CPA-security for pE, in the sense that the result we show here attests that if an encryption scheme is IND-IK-CPA-secure, then it can be safely used to construct a secure channel from an authenticated one, *while preserving anonymity.*

In the following, for a fixed encryption scheme $\Pi$ let the converter enc (where the dependency on $\Pi$ is implicit) behave as follows when connected to interface $S_i$ of $\mathsf{KEY}_{\mathcal{K}}$ and interface $S_i$ of $\mathsf{A\text{-}AUT}_{\mathcal{C}}$, for $i \in [n]$: on input a message $m \in \mathcal{M}$ from the outside, if not already done so before, output $\diamond$ to $\mathsf{KEY}_{\mathcal{K}}$ in order to fetch key $K_i$, then compute $c \leftarrow \mathtt{Enc}_{K_i}(m) \in \mathcal{C}$ and output $c$ to $\mathsf{A\text{-}AUT}_{\mathcal{C}}$. Also let the converter dec (where again the dependency on $\Pi$ is implicit) behave as follows when connected to interface $R$ of $\mathsf{KEY}_{\mathcal{K}}$ and interface $R$ of $\mathsf{A\text{-}AUT}_{\mathcal{C}}$: on input $\diamond$ from the outside, if not already done so before, output $\diamond$ to $\mathsf{KEY}_{\mathcal{K}}$ in order to fetch keys $K_1, \ldots, K_n$, and then output $\diamond$ to $\mathsf{A\text{-}AUT}_{\mathcal{C}}$; for each obtained tuple $(j, c, i)$, compute $m \leftarrow \mathtt{Dec}_{K_i}(c)$, and output the collection of all such resulting tuples $(j, m, i)$ to the outside. Finally, we define the $n$-protocol $\pi_{\mathsf{enc}} \doteq (\mathsf{enc}, \ldots, \mathsf{enc}, \mathsf{dec})$.

**Definition 14 (Composable Anonymous Security of pE).** *An encryption scheme* $\Pi$ *achieves* composable anonymous confidentiality *if*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{enc}}, \varepsilon} \mathsf{A\text{-}SEC}^n_{\mathcal{M}},$$

*that is, if there exists a simulator* sim *such that for all distinguishers* $\mathbf{D}$*,*

$$\Delta^{\mathbf{D}}(\pi_{\mathsf{enc}}[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}], \mathsf{sim}^E\, \mathsf{A\text{-}SEC}^n_{\mathcal{M}}) \leq \varepsilon(\mathbf{D}).$$

We remark that in [AHM+15] this construction step was already presented, but for a much less efficient (but statistically secure) protocol: the idea is to double the number of sender interfaces (two interfaces per user), and transmit messages bit-by-bit. More concretely, assuming $\mathcal{M} = \{0,1\}^\ell$, for some $\ell \in \mathbb{N}$, this protocol constructs $\mathsf{A\text{-}SEC}^n_{\mathcal{M}}$ from $\mathsf{A\text{-}AUT}^{2n}_{\mathcal{R}\times[\ell]}$ (and, crucially, *no* KEY resource). It works by assigning to each outside interface $S_i$, for $i \in [n]$, two interfaces $S_{i,b}$ of $\mathsf{A\text{-}AUT}^{2n}_{\mathcal{R}\times[\ell]}$, with $b \in \{0,1\}$, and transmits each message $m = (m_1, \ldots, m_\ell) \in \mathcal{M}$ as follows: first, sample some fresh uniform randomness $r \in \mathcal{R}$, for some randomness space $\mathcal{R}$, and then, for each $j \in [\ell]$, input $(r, j)$ at interface $S_{i,m_j}$ of $\mathsf{A\text{-}AUT}^{2n}_{\mathcal{R}\times[\ell]}$. Then at the receiver interface $R$, each message is reconstructed in the obvious way: upon obtaining *all* of the $\ell$ triplets $(\cdot, (r, j), (i, m_j))$, output the triplet $(\cdot, (m_1, \ldots, m_\ell), i)$ (where we are ignoring the counters, i.e., the first arguments of the triplets). This protocol is intuitively secure because for the adversary sitting at interface $E$, its view is independent of each message $m$, and moreover it can only provoke the protocol to output an invalid message at $R$ if one of the senders reuses the same randomness value $r$ for two different messages, which can be avoided by introducing *state* by the senders. Otherwise, assuming

uniform distribution over $\mathcal{R}$, this anyway happens with very small probability, that is, by a standard approximation for the birthday paradox bound, at most $q^2/|\mathcal{R}|$, where $q$ is the total of transmitted messages.

The above protocol is nevertheless clearly inefficient: considering the construction of A-AUT using a MAC scheme, for each message of size $\ell$, the underlying MAC must be invoked $\ell$ times. Here we propose a much more efficient construction by employing symmetric-key encryption, only at the cost of doubling the size of the shared secret keys. The new protocol is more efficient because now for every message only a single invocation of both the MAC and the encryption scheme are required, independently of its size.

**Theorem 6.** *If an encryption scheme* $\Pi$ *is* IND-IK-CPA*-secure, then it achieves composable anonymous confidentiality, that is,*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{enc}}, \varepsilon} \mathsf{A\text{-}SEC}^n_{\mathcal{M}},$$

*with* $\varepsilon(\mathbf{D}) \doteq \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle)$ *and reduction system* $\mathbf{C}$ *which is attached to system* $\langle \mathbf{E}_1, \ldots, \mathbf{E}_n \rangle$ *and is defined as follows.*

$\mathbf{C}\langle \mathbf{E}_1, \ldots, \mathbf{E}_n \rangle$

$\mathcal{S}, \mathcal{R} \subseteq \mathbb{N} \times \mathcal{M} \times \mathbb{N},\ c_S, c_R, t_S, t_R \in \mathbb{N}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0$
**Interface** $S_i(m)$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, m, i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, \mathbf{E}_i(m)) \in \mathbb{N} \times \mathcal{C} \mid (j, m, i) \in \mathcal{S}, c_S \leq j \leq t_S\}$
  $c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(j)$:
  **if** $\exists m \in \mathcal{M}, i \in [n] : (j, m, i) \in \mathcal{S}$ **then**
    $t_R \leftarrow t_R + 1,\ \mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, m, i)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, m, i) \in \mathcal{R} \mid c_R \leq j \leq t_R\},\ c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

*Proof.* Consider the simulator sim attached to interface $E$ of $\mathsf{A\text{-}SEC}^n_{\mathcal{M}}$ that behaves as follows: Initially, sample a key $K$ according to Gen. Then:

- On input $\diamond$ from the outside, output $\diamond$ on the inside, obtain a set $\mathcal{O} \subseteq \mathbb{N} \times \mathbb{N}$, and initialize another set $\mathcal{O}' \subseteq \mathbb{N} \times \mathcal{M}$ to $\varnothing$; Then for each $(j, \ell) \in \mathcal{O}$, add $(j, \mathtt{Enc}_K(\tilde{m}))$ to $\mathcal{O}'$, for freshly and uniformly sampled $\tilde{m} \in \mathcal{M}$ with $|\tilde{m}| = |m|$. Finally, output $\mathcal{O}'$.
- On input $c \in \mathcal{C}$ from the outside, if there exists a $j \in \mathbb{N}$ such that $(j, c) \in \mathcal{T}$, then forward $j$ to the inside.

Let now define the $n$-resources[15]

$$\mathbf{R} \doteq \mathsf{enc}^{S_1} \cdots \mathsf{enc}^{S_n}\, \mathsf{dec}^{R}\, [\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}] \quad \text{and} \quad \mathbf{S} \doteq \mathsf{sim}^{E}\, \mathsf{A\text{-}SEC}^n_{\mathcal{M}}.$$

Then we need to show that

$$\Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{S}) = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle)$$

for every distinguisher $\mathbf{D}$. For this, we first introduce the system $\mathbf{H}_0$ defined below, for which it can be easily checked that $\mathbf{R} \equiv \mathbf{H}_0$ holds ($\mathbf{H}_0$ is the monolithic representation of $\mathbf{R}$).

**$\mathbf{H}_0$**

$\mathcal{S}, \mathcal{R} \subseteq \mathbb{N} \times \mathcal{C} \times \mathbb{N},\ c_S, c_R, t_S, t_R \in \mathbb{N},\ K_1, \ldots, K_n \in \mathcal{K}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0,\ K_1, \ldots, K_n \overset{\text{iid}}{\leftarrow} \mathtt{Gen}()$
**Interface** $S_i(m)$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, \mathtt{Enc}_{K_i}(m), i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, c) \in \mathbb{N} \times \mathcal{C} \mid \exists i \in [n] : (j, c, i) \in \mathcal{S}, c_S \leq j \leq t_S\},\ c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(j)$:
  **if** $\exists c \in \mathcal{C}, i \in [n] : (j, c, i) \in \mathcal{S}$ **then**
    $t_R \leftarrow t_R + 1,\ \mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, c, i)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, \mathtt{Dec}_{K_i}(c), i) \in \mathbb{N} \times \mathcal{M} \times [n] \mid (j, c, i) \in \mathcal{R}, c_R \leq j \leq t_R\}$
  $c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

We now define the system $\mathbf{H}_1 \doteq \mathbf{C}[\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}]$, for which it can be easily checked that $\mathbf{H}_0 \equiv \mathbf{H}_1$ holds (by the correctness of the scheme). Next, we define the system $\mathbf{H}_2 \doteq \mathbf{C}\langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle$, for which it can be easily checked that $\mathbf{H}_2 \equiv \mathbf{S}$ holds ($\mathbf{H}_2$ is the monolithic representation of $\mathbf{S}$). Summarizing, we established that

$$\mathbf{R} \equiv \mathbf{H}_0 \equiv \mathbf{H}_1 \quad \text{and} \quad \mathbf{H}_2 \equiv \mathbf{S},$$

and therefore by Lemma 2 we have

$$\begin{aligned}
\Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{S}) &= \Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{H}_0) + \Delta^{\mathbf{D}}(\mathbf{H}_0, \mathbf{H}_1) + \Delta^{\mathbf{D}}(\mathbf{H}_1, \mathbf{H}_2) + \Delta^{\mathbf{D}}(\mathbf{H}_2, \mathbf{S}) \\
&= 0 + 0 + \Delta^{\mathbf{D}}(\mathbf{C}[\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \mathbf{C}\langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle) + 0 \\
&= \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle) \\
&= \varepsilon(\mathbf{D}),
\end{aligned}$$

[15] For simplicity, here we consider the slightly different channel resources which on input $-1$ at interface $E$ do nothing (instead of adding the tuple $(k, \perp, \perp)$, for some $k \in \mathbb{N}$, to the set $\mathcal{R}$), since they would behave identically also otherwise.

which indeed implies

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{enc}}, \varepsilon} \mathsf{A\text{-}SEC}^n_{\mathcal{M}}. \qquad \square$$

Note that by Theorem 1, it must be possible to prove Theorem 6 with $\varepsilon'(\mathbf{D}) = f(n) \cdot \Delta^{\mathbf{DC}'}(\mathbf{E}_K, \$)$, for some polynomial $f(n)$ and (different) reduction system $\mathbf{C}'$, rather than $\varepsilon(\mathbf{D}) = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle)$. More precisely, we remark that by virtue of Lemma 5, Lemma 6 and Lemma 7 for $n = 1$, and Theorem 1, for appropriate distinguishers $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}_4$,

$$\begin{aligned}
\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}^{\$}_K, \ldots, \mathbf{E}^{\$}_K \rangle) &= (n-1) \cdot \Delta^{\mathbf{D}_1}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K, \mathbf{E}_K \rangle) \\
&\quad + \Delta^{\mathbf{D}_2}(\mathbf{E}_K, \mathbf{E}^{\$}_K) \\
&= (2n-1) \cdot \Delta^{\mathbf{D}_3}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}^{\$}_K, \mathbf{E}^{\$}_K \rangle) \\
&= (6n-3) \cdot \Delta^{\mathbf{D}_4}(\mathbf{E}_K, \$).
\end{aligned}$$

Therefore, by letting $\mathbf{C}'$ be the reduction system resulting from the composition of the various reduction systems from the mentioned results, we have that, for $\mathbf{R}$ and $\mathbf{S}$ as in the proof of Theorem 6,

$$\Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{S}) = (6n-3) \cdot \Delta^{\mathbf{DC}'}(\mathbf{E}_K, \$),$$

which implies

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{enc}}, \varepsilon'} \mathsf{A\text{-}SEC}^n_{\mathcal{M}},$$

for $f(n) = 6n - 3$.

### 4.4 Composable Anonymous Security of pAE

In this section we first introduce a composable definition of anonymous security for pAE, and then we show that the previously introduced game-based notion of IND-IK-CCA3-security implies the former. The composable definition can be interpreted as providing *composable semantics* to IND-IK-CCA3-security for pAE, in the sense that the result we show here attests that if an (authenticated) encryption scheme is IND-IK-CCA3-secure, then it can be safely used to construct a secure channel from an insecure one, *while preserving anonymity.*

In the following, for a fixed (authenticated) encryption scheme $\Pi$ let the converter ae (where the dependency on $\Pi$ is implicit) behave as follows when connected to interface $S_i$ of $\mathsf{KEY}_{\mathcal{K}}$ and interface $S_i$ of $\mathsf{A\text{-}INS}_{\mathcal{C}}$, for $i \in [n]$: on input a message $m \in \mathcal{M}$ from the outside, if not already done so before, output $\diamond$ to $\mathsf{KEY}_{\mathcal{K}}$ in order to fetch key $K_i$, then compute $c \leftarrow \mathtt{Enc}_{K_i}(m) \in \mathcal{C}$ and output $c$ to $\mathsf{A\text{-}INS}_{\mathcal{C}}$. Also let the converter ad (where again the dependency on $\Pi$ is implicit) behave as follows when connected to interface $R$ of $\mathsf{KEY}_{\mathcal{K}}$ and interface $R$ of $\mathsf{A\text{-}INS}_{\mathcal{C}}$: on input $\diamond$ from the outside, if not already done so before, output $\diamond$ to $\mathsf{KEY}_{\mathcal{K}}$ in order to fetch keys $K_1, \ldots, K_n$, and then output $\diamond$ to $\mathsf{A\text{-}INS}_{\mathcal{C}}$; for each obtained tuple $(j, c)$, find the index $i \in [n]$ such that $m \neq \bot$, for $m \leftarrow \mathtt{Dec}_{K_i}(c)$, and output the collection of all such resulting tuples $(j, m, i)$ to the outside. Finally, we define the $n$-protocol $\pi_{\mathsf{ae}} \doteq (\mathsf{ae}, \ldots, \mathsf{ae}, \mathsf{ad})$.

**Definition 15 (Composable Anonymous Security of pAE).** *An (authenticated) encryption scheme* $\Pi$ *achieves* composable anonymous security *if*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{ae}}, \varepsilon} \mathsf{A\text{-}SEC}^n_{\mathcal{M}},$$

*that is, if there exists a simulator* $\mathsf{sim}$ *such that for all distinguishers* $\mathbf{D}$*,*

$$\Delta^{\mathbf{D}}(\pi_{\mathsf{ae}}[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}}], \mathsf{sim}^E\, \mathsf{A\text{-}SEC}^n_{\mathcal{M}}) \leq \varepsilon(\mathbf{D}).$$

Again, we remark that in [AHM+15, Theorem 2] this direct construction step was already presented but the suggested protocol is again much less efficient than ours. The idea improves upon the previous one used to construct $\mathsf{A\text{-}SEC}^n_{\mathcal{M}}$ from $\mathsf{A\text{-}AUT}^{2n}_{\mathcal{R}\times[\ell]}$, by using the randomness $r \in \mathcal{R}$ only once per message, and reducing the domain of the underlying MAC scheme to $|\mathcal{R}| + \log \ell$ bits (where again we are assuming $\mathcal{M} = \{0,1\}^{\ell}$). Detailedly, given a MAC with message space $\mathcal{M}' \doteq \mathcal{R} \times \{0,1\}^{\log \ell}$ and tag space $\mathcal{T}$, the protocol uses $[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{R}\times\mathcal{T}^{\ell}}]$ in the following way: on input a message $m = (m_1, \ldots, m_{\ell}) \in \mathcal{M}$ at the outside interface assigned to sender $S_i$, compute $c \doteq (r, \mathtt{Tag}_{k_{i,m_1}}(r,1), \ldots, \mathtt{Tag}_{k_{i,m_{\ell}}}(r,\ell))$, where $r$ is sampled uniformly at random over $\mathcal{R}$, $k_{i,0}$ is the key shared by $S_i$ and $R$ through the first $\mathsf{KEY}^n_{\mathcal{K}}$ resource, and $k_{i,1}$ is the key shared by $S_i$ and $R$ through the second $\mathsf{KEY}^n_{\mathcal{K}}$ resource. Then at the receiver interface $R$, each message is reconstructed by testing the value $(r, \tau_1, \ldots, \tau_{\ell})$ obtained by $\mathsf{A\text{-}INS}^n_{\mathcal{R}\times\mathcal{T}^{\ell}}$ against each possible key-pair $(k_{i,0}, k_{i,1})$, for $i \in [n]$, and message $(m_1, \ldots, m_{\ell}) \in \{0,1\}^{\ell}$: if for each $j \in [\ell]$ the tag $\tau_j$ is valid for the (MAC) message $(r, j) \in \mathcal{R} \times \{0,1\}^{\log \ell}$ under key $k_{i,m_j}$, then output $(\cdot, (m_1, \ldots, m_{\ell}), i)$

Note that the major drawbacks of this construction are (1) the fact that even if the message space of the MAC has been reduced, this must be invoked $\ell$ times for each message (as opposed to 1 time), and (2) the fact that the time complexity of the receiver is $\mathcal{O}(n\ell)$ for each message (as opposed to $\mathcal{O}(n)$). Here we improve the efficiency of this construction by employing authenticated encryption instead; therefore, this can be seen as improving upon both the amount of invocations to the underlying primitive (once per message—once MAC and once encryption, if the scheme arises from the Encrypt-then-MAC paradigm—instead of $\ell$), and the time complexity associated to the receiving of each message: we only need to test the received ciphertext against each possible of the $n$ keys. Moreover, our construction statement arguably feels more "natural" than the one of [AHM+15].

**Theorem 7.** *If an (authenticated) encryption scheme* $\Pi$ *is* IND-IK-CCA3*-secure, then it achieves composable anonymous security, that is,*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}}] \xrightarrow{\pi_{\mathsf{ae}}, \varepsilon} \mathsf{A\text{-}SEC}^n_{\mathcal{M}},$$

*with* $\varepsilon(\mathbf{D}) \doteq \Delta^{\mathbf{DC}}([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1} \rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n} \rangle], \langle \langle \mathbf{E}^{\$}_K, \mathbf{D}^{\perp} \rangle, \ldots, \langle \mathbf{E}^{\$}_K, \mathbf{D}^{\perp} \rangle \rangle)$ *and reduction system* $\mathbf{C}$ *which is attached to system* $\langle \langle \mathbf{E}_1, \mathbf{D}_1 \rangle, \ldots, \langle \mathbf{E}_n, \mathbf{D}_n \rangle \rangle$ *and is defined as follows.*

$\mathbf{C}\langle\langle \mathbf{E}_1, \mathbf{D}_1\rangle, \ldots, \langle \mathbf{E}_n, \mathbf{D}_n\rangle\rangle$

$\mathcal{S}, \mathcal{R} \subseteq (\mathbb{N} \times \mathcal{M} \times \mathbb{N}) \cup (\mathbb{N} \times \{\bot\}^2),\ c_S, c_R, t_S, t_R \in \mathbb{N}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0$
**Interface** $S_i(m)$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, m, i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, \mathbf{E}_i(m)) \in \mathbb{N} \times \mathcal{C} \mid (j, m, i) \in \mathcal{S}, c_S \leq j \leq t_S\}$
  $c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(c)$:
  $t_R \leftarrow t_R + 1$
  **if** $\exists i \in [n] : \mathbf{D}_i(c) \neq \bot$ **then**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, \mathbf{D}_i(c), i)\}$
  **else**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, \bot, \bot)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, m, i) \in \mathcal{R} \mid c_R \leq j \leq t_R\},\ c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

*Proof.* Consider the simulator $\mathsf{sim}$ attached to interface $E$ of $\mathsf{A\text{-}SEC}^n_{\mathcal{M}}$ that behaves as follows: Initially, sample a key $K$ according to $\mathtt{Gen}$, and initialize the set $\mathcal{T}$ to $\varnothing$. Then:

- On input $\diamond$ from the outside, output $\diamond$ on the inside, obtain a set $\mathcal{O} \subseteq \mathbb{N} \times \mathbb{N}$, and initialize another set $\mathcal{O}' \subseteq \mathbb{N} \times \mathcal{M}$ to $\varnothing$; Then for each $(j, \ell) \in \mathcal{O}$, add $(j, \mathtt{Enc}_K(\tilde{m}))$ to both $\mathcal{O}'$ and $\mathcal{T}$, for freshly and uniformly sampled $\tilde{m} \in \mathcal{M}$ with $|\tilde{m}| = |m|$. Finally, output $\mathcal{O}'$.
- On input $j \in \mathbb{N}$ from the outside, simply forward $j$ to the inside.

Let now define the $n$-resources

$$\mathbf{R} \doteq \mathsf{ae}^{S_1} \cdots \mathsf{ae}^{S_n}\, \mathsf{ad}^R\, [\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}}] \quad \text{and} \quad \mathbf{S} \doteq \mathsf{sim}^E\, \mathsf{A\text{-}SEC}^n_{\mathcal{M}}.$$

Then we need to show that

$$\Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{S}) = \Delta^{\mathbf{DC}}([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1}\rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n}\rangle], [\langle \mathbf{E}^{\$}_K, \mathbf{D}^{\bot}\rangle, \ldots, \langle \mathbf{E}^{\$}_K, \mathbf{D}^{\bot}\rangle])$$

for every distinguisher $\mathbf{D}$. For this, we first introduce the system $\mathbf{H}_0$ defined below, for which it can be easily checked that $\mathbf{R} \equiv \mathbf{H}_0$ holds ($\mathbf{H}_0$ is the monolithic representation of $\mathbf{R}$).

**$\mathbf{H}_0$**

$\mathcal{S}, \mathcal{R} \subseteq \mathbb{N} \times \mathcal{C},\ c_S, c_R, t_S, t_R \in \mathbb{N},\ K_1, \ldots, K_n \in \mathcal{K}$
**Initialize**:
  $\mathcal{S}, \mathcal{R} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0,\ K_1, \ldots, K_n \xleftarrow{\text{iid}} \mathtt{Gen}()$
**Interface** $S_i(m)$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, \mathtt{Enc}_{K_i}(m))\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, c) \in \mathcal{S} \mid c_S \leq j \leq t_S\},\ c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(c)$:
  $t_R \leftarrow t_R + 1,\ \mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, c)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, \mathtt{Dec}_{K_i}(c), i) \in \mathbb{N} \times \mathcal{M} \times [n] \mid (j, c) \in \mathcal{R}, c_R \leq j \leq t_R, \mathtt{Dec}_{K_i}(c) \neq \perp\}$
    $\cup \{(j, \perp, \perp) \mid \exists c \in \mathcal{C} : (j, c) \in \mathcal{R}, c_R \leq j \leq t_R, \forall i \in [n] : \mathtt{Dec}_{K_i}(c) = \perp\}$
  $c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

We now introduce the system $\mathbf{H}_1 \doteq \mathbf{C}[\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1} \rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n} \rangle]$, for which it can be easily checked that $\mathbf{H}_0 \equiv \mathbf{H}_1$ holds (this is just a different description of the same system). Next, we introduce the system $\mathbf{H}_3$ defined below, for which it can be easily checked that $\mathbf{H}_3 \equiv \mathbf{S}$ holds ($\mathbf{H}_3$ is the monolithic representation of $\mathbf{S}$).

**$\mathbf{H}_3$**

$\mathcal{S}, \mathcal{R} \subseteq (\mathbb{N} \times \mathcal{M} \times \mathbb{N}) \cup (\mathbb{N} \times \{\perp\}^2),\ \mathcal{T} \subseteq \mathbb{N} \times \mathcal{C},\ c_S, c_R, t_S, t_R \in \mathbb{N},\ K \in \mathcal{K}$
**Initialize**:
  $\mathcal{S}, \mathcal{R}, \mathcal{T} \leftarrow \varnothing,\ c_S, c_R \leftarrow 1,\ t_S, t_R \leftarrow 0,\ K \leftarrow \mathtt{Gen}()$
**Interface** $S_i(m)$:
  $t_S \leftarrow t_S + 1,\ \mathcal{S} \leftarrow \mathcal{S} \cup \{(t_S, m, i)\}$
**Interface** $E(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, \mathtt{Enc}_K(\$^{|m|})) \in \mathbb{N} \times \mathcal{C} \mid \exists i \in [n] : (j, m, i) \in \mathcal{S}, c_S \leq j \leq t_S\}$
  $\mathcal{T} \leftarrow \mathcal{T} \cup \mathcal{O},\ c_S \leftarrow t_S + 1$
  **return** $\mathcal{O}$
**Interface** $E(c)$:
  $t_R \leftarrow t_R + 1$
  **if** $\exists j \in \mathbb{N} : (j, c) \in \mathcal{T}$ **then**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, m, i) \in \mathbb{N} \times \mathcal{M} \times \mathbb{N} \mid (j, m, i) \in \mathcal{S}\}$
  **else**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(t_R, \perp, \perp)\}$
**Interface** $R(\diamond)$:
  $\mathcal{O} \leftarrow \{(j, m, i) \in \mathcal{R} \mid c_R \leq j \leq t_R\},\ c_R \leftarrow t_R + 1$
  **return** $\mathcal{O}$

Finally, we introduce the system $\mathbf{H}_2 \doteq \mathbf{C}[\langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp} \rangle, \ldots, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp} \rangle]$, for which it can be easily checked that $\mathbf{H}_2 \equiv \mathbf{H}_3$ holds (by the correctness of the scheme). Summarizing, we established that

$$\mathbf{R} \equiv \mathbf{H}_0 \equiv \mathbf{H}_1 \quad \text{and} \quad \mathbf{H}_2 \equiv \mathbf{H}_3 \equiv \mathbf{S},$$

and therefore by Lemma 2 we have

$$\begin{aligned}
\Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{S}) &= \Delta^{\mathbf{D}}(\mathbf{R}, \mathbf{H}_0) + \sum_{i=0}^{2} \Delta^{\mathbf{D}}(\mathbf{H}_i, \mathbf{H}_{i+1}) + \Delta^{\mathbf{D}}(\mathbf{H}_3, \mathbf{S}) \\
&= 0 + 0 + \Delta^{\mathbf{D}}(\mathbf{H}_1, \mathbf{H}_2) + 0 + 0 \\
&= \Delta^{\mathbf{DC}}([\langle \mathbf{E}_{K_1}, \mathbf{D}_{K_1} \rangle, \ldots, \langle \mathbf{E}_{K_n}, \mathbf{D}_{K_n} \rangle], [\langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp} \rangle, \ldots, \langle \mathbf{E}_K^{\$}, \mathbf{D}^{\perp} \rangle]) \\
&= \varepsilon(\mathbf{D}),
\end{aligned}$$

which indeed implies

$$[\mathsf{KEY}_{\mathcal{K}}^{n}, \mathsf{A\text{-}AUT}_{\mathcal{C}}^{n}] \xrightarrow{\pi_{\mathsf{ae}}, \varepsilon} \mathsf{A\text{-}SEC}_{\mathcal{M}}^{n}. \qquad \square$$

Note that, analogously as for $\mathsf{pE}$, by virtue of Corollary 2, Corollary 3 and Corollary 4 for $n = 1$, and Theorem 2, it is possible to prove Theorem 7 with

$$\varepsilon'(\mathbf{D}) = (6n - 3) \cdot \Delta^{\mathbf{DC}'}(\langle \mathbf{E}_K, \mathbf{D}_K \rangle, \langle \$, \mathbf{D}^{\perp} \rangle),$$

for an appropriate reduction system $\mathbf{C}'$.

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## A Proofs for Section 2

**Lemma 1.** *For distinguisher* $\mathbf{D}$ *and systems* $\mathbf{S}$ *and* $\mathbf{T}$, $\Delta^{\mathbf{D}}(\mathbf{S},\mathbf{T}) = \Delta^{\mathbf{DI}}(\mathbf{T},\mathbf{S})$.

*Proof.*

$$\begin{aligned}\Delta^{\mathbf{D}}(\mathbf{S},\mathbf{T}) &= \Pr[\mathbf{DS}=0] - \Pr[\mathbf{DT}=0] \\ &= \Pr[\mathbf{DIS}=1] - \Pr[\mathbf{DIT}=1] \\ &= 1 - \Pr[\mathbf{DIS}=0] - 1 + \Pr[\mathbf{DIT}=0] \\ &= \Pr[\mathbf{DIT}=0] - \Pr[\mathbf{DIS}=0] \\ &= \Delta^{\mathbf{DI}}(\mathbf{T},\mathbf{S}).\end{aligned}$$

□

**Lemma 2.** *For distinguisher* $\mathbf{D}$ *and systems* $\mathbf{S}_1,\ldots,\mathbf{S}_n$,

$$\Delta^{\mathbf{D}}(\mathbf{S}_1,\mathbf{S}_n) = \sum_{i=1}^{n-1} \Delta^{\mathbf{D}}(\mathbf{S}_i,\mathbf{S}_{i+1}).$$

*Proof.*

$$\begin{aligned}\sum_{i=1}^{n-1} \Delta^{\mathbf{D}}(\mathbf{S}_i,\mathbf{S}_{i+1}) &= \Pr[\mathbf{DS}_1=0] - \Pr[\mathbf{DS}_2=0] + \cdots \\ &\qquad + \Pr[\mathbf{DS}_{n-1}=0] - \Pr[\mathbf{DS}_n=0] \\ &= \Pr[\mathbf{DS}_1=0] - \Pr[\mathbf{DS}_n=0] \\ &= \Delta^{\mathbf{D}}(\mathbf{S}_1,\mathbf{S}_n).\end{aligned}$$

□

**Lemma 3.** *For distinguishers* $\mathbf{D}_1,\ldots,\mathbf{D}_n$, *systems* $\mathbf{S}$ *and* $\mathbf{T}$, *and random variable* $I$ *uniformly distributed over* $[n]$,

$$\sum_{i=1}^{n} \Delta^{\mathbf{D}_i}(\mathbf{S},\mathbf{T}) = n \cdot \Delta^{\mathbf{D}_I}(\mathbf{S},\mathbf{T}).$$

*Proof.* By the law of total probability:

$$\begin{aligned}\Delta^{\mathbf{D}_I}(\mathbf{S},\mathbf{T}) &= \Pr[\mathbf{D}_I\mathbf{S}=0] - \Pr[\mathbf{D}_I\mathbf{T}=0] \\ &= \sum_{i=1}^{n} \big(\Pr[\mathbf{D}_i\mathbf{S}=0]\cdot\Pr[I=i] - \Pr[\mathbf{D}_i\mathbf{T}=0]\cdot\Pr[I=i]\big) \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n} \big(\Pr[\mathbf{D}_i\mathbf{S}=0] - \Pr[\mathbf{D}_i\mathbf{T}=0]\big) \\ &= \frac{1}{n}\cdot\sum_{i=1}^{n} \Delta^{\mathbf{D}_i}(\mathbf{S},\mathbf{T}).\end{aligned}$$

□

## B Proofs for Section 3

**Lemma 4.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\dots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K,\dots,\mathbf{E}_K\rangle) = (n-1)\cdot\Delta^{\mathbf{DC}}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle).$$

*In particular, this implies that if an encryption scheme is* 2-IK-CPA*-secure, then it is also* $n$-IK-CPA*-secure.*

*Proof.* For $i \in [n]$, let define hybrid systems

$$\mathbf{H}_i \doteq \langle\underbrace{\mathbf{E}_K,\dots,\mathbf{E}_K}_{i \text{ times}},\mathbf{E}_{K_{i+1}},\dots,\mathbf{E}_{K_n}\rangle$$

and reduction systems $\mathbf{C}_i$ such that

$$\mathbf{C}_i\langle\mathbf{S},\mathbf{T}\rangle = \langle\underbrace{\mathbf{S},\dots,\mathbf{S}}_{i \text{ times}},\mathbf{T},\mathbf{E}_{K_{i+2}},\dots,\mathbf{E}_{K_n}\rangle.$$

Then note that $\mathbf{H}_i = \mathbf{C}_i[\mathbf{E}_{K_1},\mathbf{E}_{K_2}]$ and $\mathbf{H}_{i+1} = \mathbf{C}_i\langle\mathbf{E}_K,\mathbf{E}_K\rangle$. Therefore, for any distinguisher $\mathbf{D}$, by Lemma 2 and Lemma 3,

$$\begin{aligned}
\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\dots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K,\dots,\mathbf{E}_K\rangle) &= \sum_{i=1}^{n-1}\Delta^{\mathbf{D}}(\mathbf{H}_i,\mathbf{H}_{i+1})\\
&= \sum_{i=1}^{n-1}\Delta^{\mathbf{D}}(\mathbf{C}_i[\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\mathbf{C}_i\langle\mathbf{E}_K,\mathbf{E}_K\rangle)\\
&= \sum_{i=1}^{n-1}\Delta^{\mathbf{DC}_i}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle)\\
&= (n-1)\cdot\Delta^{\mathbf{DC}_I}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle).
\end{aligned}$$

where $I$ is uniformly distributed over $[n]$ With $\mathbf{C} \doteq \mathbf{C}_I$, this concludes the proof. □

**Lemma 5.** *For every distinguisher* $\mathbf{D}$*, there exist reductions* $\mathbf{C}$ *and* $\mathbf{C}'$ *such that*

$$\begin{aligned}
\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\dots,\mathbf{E}_{K_n}],\langle\mathbf{E}^{\$}_K,\dots,\mathbf{E}^{\$}_K\rangle) = (n-1)\cdot\Delta^{\mathbf{DC}}([\mathbf{E}_{K_1},\mathbf{E}_{K_2}],\langle\mathbf{E}_K,\mathbf{E}_K\rangle)\\
+\Delta^{\mathbf{DC}'}(\mathbf{E}_K,\mathbf{E}^{\$}_K).
\end{aligned}$$

*In particular, this implies that if an encryption scheme is* 2-IK-CPA*-secure and* IND-CPA*-secure, then it is also* $n$-IND-IK-CPA*-secure.*

*Proof.* First note that by Lemma 2, for any distinguisher $\mathbf{D}$,

$$\begin{aligned}
\Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\dots,\mathbf{E}_{K_n}],\langle\mathbf{E}^{\$}_K,\dots,\mathbf{E}^{\$}_K\rangle) = \Delta^{\mathbf{D}}([\mathbf{E}_{K_1},\dots,\mathbf{E}_{K_n}],\langle\mathbf{E}_K,\dots,\mathbf{E}_K\rangle)\\
+\Delta^{\mathbf{D}}(\langle\mathbf{E}_K,\dots,\mathbf{E}_K\rangle,\langle\mathbf{E}^{\$}_K,\dots,\mathbf{E}^{\$}_K\rangle).
\end{aligned}$$

Moreover, for reduction system $\mathbf{C}'$ such that $\mathbf{C}'\mathbf{S} = \langle \underbrace{\mathbf{S}, \ldots, \mathbf{S}}_{n \text{ times}} \rangle$, then

$$\Delta^{\mathbf{D}}(\langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle, \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) = \Delta^{\mathbf{D}}(\mathbf{C}'\mathbf{E}_K, \mathbf{C}'\mathbf{E}_K^{\$}) = \Delta^{\mathbf{DC}'}(\mathbf{E}_K, \mathbf{E}_K^{\$}).$$

With $\mathbf{C}$ as defined in the proof of Lemma 4, this concludes the proof. □

**Lemma 6.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K, \mathbf{E}_K \rangle) = 2 \cdot \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle).$$

*In particular, this implies that if an encryption scheme is* $n$-IND-IK-CPA*-secure, then it is also* 2-IK-CPA*-secure.*

*Proof.* For reduction system $\mathbf{C}_1$ such that $\mathbf{C}_1\langle \mathbf{S}_1, \ldots, \mathbf{S}_n \rangle = \langle \mathbf{S}_1, \mathbf{S}_2 \rangle$ and reduction system $\mathbf{C}_2$ such that $\mathbf{C}_2\langle \mathbf{S}_1, \ldots, \mathbf{S}_n \rangle = \langle \mathbf{S}_1, \ldots, \mathbf{S}_1 \rangle$, for any distinguisher $\mathbf{D}$, by Lemma 2, Lemma 1, and Lemma 3,

$$\begin{aligned}
\Delta^{\mathbf{D}}([\mathbf{E}_{K_1}, \mathbf{E}_{K_2}], \langle \mathbf{E}_K, \mathbf{E}_K \rangle) &= \Delta^{\mathbf{D}}(\mathbf{C}_1[\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \mathbf{C}_1\langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle) \\
&= \Delta^{\mathbf{DC}_1}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle) \\
&= \Delta^{\mathbf{DC}_1}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&\quad + \Delta^{\mathbf{DC}_1}(\langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle, \langle \mathbf{E}_K, \ldots, \mathbf{E}_K \rangle) \\
&= \Delta^{\mathbf{DC}_1}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&\quad + \Delta^{\mathbf{DIC}_1}(\mathbf{C}_2[\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \mathbf{C}_2\langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&= \Delta^{\mathbf{DC}_1}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&\quad + \Delta^{\mathbf{DIC}_1\mathbf{C}_2}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&= 2 \cdot \Delta^{\mathbf{DC}'_I}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle),
\end{aligned}$$

where $\mathbf{C}'_1 \doteq \mathbf{C}_1$, $\mathbf{C}'_2 \doteq \mathbf{I}\mathbf{C}_1\mathbf{C}_2$, and $I$ is uniformly distributed over $\{1, 2\}$. With $\mathbf{C} \doteq \mathbf{C}'_I$, this concludes the proof. □

**Lemma 7.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\Delta^{\mathbf{D}}(\mathbf{E}_K, \mathbf{E}_K^{\$}) = \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle).$$

*In particular, this implies that if an encryption scheme is* $n$-IND-IK-CPA*-secure, then it is also* IND-CPA*-secure.*

*Proof.* For reduction system $\mathbf{C}$ such that $\mathbf{C}\langle \mathbf{S}_1, \ldots, \mathbf{S}_n \rangle = \mathbf{S}_1$, for any distinguisher $\mathbf{D}$,

$$\begin{aligned}
\Delta^{\mathbf{D}}(\mathbf{E}_K, \mathbf{E}_K^{\$}) &= \Delta^{\mathbf{D}}(\mathbf{C}[\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \mathbf{C}\langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle) \\
&= \Delta^{\mathbf{DC}}([\mathbf{E}_{K_1}, \ldots, \mathbf{E}_{K_n}], \langle \mathbf{E}_K^{\$}, \ldots, \mathbf{E}_K^{\$} \rangle).
\end{aligned}$$

This concludes the proof. □

## C Anonymous Security of Probabilistic MACs (pMAC)

We introduce a very specific syntax for *Message Authentication Codes (MAC)* which will turn out to be very useful in order to analyze the Encrypt-then-MAC paradigm. More precisely, we consider MAC schemes which take as messages ciphertexts arising from some encryption scheme, and which provide an interface optimized for being coupled with such scheme. In this section we revisit the security and anonymity notions of MAC, the latter having being originally introduced in [AHM+14] (as a form of *key-indistinguishability*), and used in [AHM+15] to construct an authenticated and anonymous channel. Note that since we are interested in anonymity in this work, it is imperative that we only consider probabilistic MAC (pMAC), as pointed out in [AHM+14,AHM+15].

**Definition 16 (MAC Scheme).** *A* (probabilistic) message authentication code (MAC) scheme $\Sigma \doteq (\mathtt{Gen}, \mathtt{Tag}, \mathtt{Vrf})$ *over key-space* $\mathcal{K}$*, message-space* $\mathcal{C}$*, and tag-space* $\mathcal{T}$ *(with* $\perp \notin \mathcal{K} \cup \mathcal{C} \cup \mathcal{T}$*), is such that*

- $\mathtt{Gen}$ *is a distribution (often the uniform one) over* $\mathcal{K}$*;*
- $\mathtt{Tag} : \mathcal{K} \times \mathcal{C} \to \mathcal{C} \times \mathcal{T}$ *is a* probabilistic *function;*
- $\mathtt{Vrf} : \mathcal{K} \times \mathcal{C} \times \mathcal{T} \to \mathcal{C} \cup \{\perp\}$ *is a* deterministic *function.*

*As customary, for* $k \in \mathcal{K}$ *we use the short-hand notation* $\mathtt{Tag}_k(\cdot)$ *for* $\mathtt{Tag}(k, \cdot)$ *and* $\mathtt{Vrf}_k(\cdot, \cdot)$ *for* $\mathtt{Vrf}(k, \cdot, \cdot)$*. Moreover, we assume* correctness *of* $\Sigma$*, that is, for all keys* $k$ *distributed according to* $\mathtt{Gen}$*, and all ciphertext-tag pairs* $(c, \tau) \in \mathcal{C} \times \mathcal{T}$*,*

$$\mathtt{Vrf}_k(c, \tau) = \begin{cases} c & \text{if } (c, \tau) \in \operatorname{supp}(\mathtt{Tag}_k(c)), \\ \perp & \text{otherwise.} \end{cases}$$

As for pE and pAE, in order to define the security and anonymity of a fixed MAC scheme $\Sigma$, we need to define the following single and double interface systems (where the dependency on $\Sigma$ is implicit), parameterized by a fixed key $k \in \mathcal{K}$:

- $\langle \mathbf{T}_k, \mathbf{V}_k \rangle$:
  - On input a ciphertext $c \in \mathcal{C}$, return $\mathtt{Tag}_k(c) \in \mathcal{C} \times \mathcal{T}$.
  - On input a ciphertext-tag pair $(c, \tau) \in \mathcal{C} \times \mathcal{T}$, return $\mathtt{Vrf}_k(c, \tau) \in \mathcal{C} \cup \{\perp\}$.
- $\langle \mathbf{T}_k, \mathbf{V}^{\perp} \rangle$: Initially set $\mathcal{Q} \subseteq \mathcal{C} \times \mathcal{T}$ to $\varnothing$ and then:
  - On input a ciphertext $c \in \mathcal{C}$, return $(c, \tau) \doteq \mathtt{Tag}_k(c) \in \mathcal{C} \times \mathcal{T}$ and set $\mathcal{Q}$ to $\mathcal{Q} \cup \{(c, \tau)\}$.
  - On input a ciphertext-tag pair $(c, \tau) \in \mathcal{C} \times \mathcal{T}$, if $(c, \tau) \in \mathcal{Q}$ then return $c$, otherwise return $\perp$.

In our definitions, the key $k$ will *always* be replaced by a random variable (usually denoted $K$ or $K_i$, for some $i \in \mathbb{N}$) distributed according to $\Sigma$'s $\mathtt{Gen}$.

### C.1 Game-Based (Anonymous) Security of pMAC

The classical security notion of MAC is *universal unforgeability under chosen messages attack*. This kind of game-based definition is often formulated as a game which an adversary is supposed to win. In this work we take the dual view that such a definition can be equivalently phrased as a distinction problem (see for example [Mau02,MPR07,Ros18]).

**Definition 17 (Game-Based Unforgeability of pMAC).** *A MAC scheme* $\Sigma$ *is* unforgeable pMAC *(or* UF-CMA-secure*) if*

$$\Delta^{\mathbf{D}}(\langle \mathbf{T}_K, \mathbf{V}_K \rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

The concept of anonymous MAC schemes was crystallized by Alwen et al, which in [AHM+14] introduced the notion of key-indistinguishable pMAC. In the following definition, we capture this notion within our framework.

**Definition 18 (Game-Based Anonymity of pMAC).** *A MAC scheme* $\Sigma$ *is* [$n$-]anonymous pMAC *(or* [$n$-]IK-CMA-secure*) if*

$$\Delta([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1} \rangle, \ldots, \langle \mathbf{T}_{K_n}, \mathbf{V}_{K_n} \rangle], \langle \langle \mathbf{T}_K, \mathbf{V}_K \rangle, \ldots, \langle \mathbf{T}_K, \mathbf{V}_K \rangle \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

Finally, we introduce a new all-in-one definition for pMAC, which should intuitively capture both unforgeability and anonymity.

**Definition 19 (Game-Based Anonymous Unforgeability of pMAC).** *A MAC scheme* $\Sigma$ *is* [$n$-]anonymous secure pMAC *(or* [$n$-]UF-IK-CMA-secure*) if*

$$\Delta([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1} \rangle, \ldots, \langle \mathbf{T}_{K_n}, \mathbf{V}_{K_n} \rangle], \langle \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle, \ldots, \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle \rangle)$$

*is negligible for any efficient distinguisher* $\mathbf{D}$*.*

### C.2 Relations Among Notions

We now confirm the intuition that UF-CMA and IK-CMA together imply UF-IK-CMA (for the simpler case of $n + 2$, which is easily generalizable).

**Lemma 8.** *For every distinguisher* $\mathbf{D}$*, there exists a reduction* $\mathbf{C}$ *such that*

$$\begin{aligned}
&\Delta^{\mathbf{D}}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1} \rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2} \rangle], \langle \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle \rangle) \\
&\quad = \Delta^{\mathbf{D}}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1} \rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2} \rangle], \langle \langle \mathbf{T}_K, \mathbf{V}_K \rangle, \langle \mathbf{T}_K, \mathbf{V}_K \rangle \rangle) \\
&\qquad + \Delta^{\mathbf{DC}}(\langle \mathbf{T}_K, \mathbf{V}_K \rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp} \rangle).
\end{aligned}$$

*In particular, this implies that if a MAC scheme is* UF-CMA*-secure and* IK-CMA*-secure, then it is also* UF-IK-CMA*-secure.*

*Proof.* First note that by Lemma 2, for any distinguisher $\mathbf{D}$,

$$\begin{aligned}
&\Delta^{\mathbf{D}}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2}\rangle], \langle\langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle\rangle) \\
&\quad = \Delta^{\mathbf{D}}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \langle \mathbf{T}_{K_2}, \mathbf{V}_{K_2}\rangle], \langle\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \langle \mathbf{T}_K, \mathbf{V}_K\rangle\rangle) \\
&\qquad + \Delta^{\mathbf{D}}(\langle\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \langle \mathbf{T}_K, \mathbf{V}_K\rangle\rangle, \langle\langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle\rangle).
\end{aligned}$$

Moreover, for reduction system $\mathbf{C}$ such that $\mathbf{CS} = \langle \mathbf{S}, \mathbf{S}\rangle$, then

$$\begin{aligned}
&\Delta^{\mathbf{D}}(\langle\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \langle \mathbf{T}_K, \mathbf{V}_K\rangle\rangle, \langle\langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle\rangle) \\
&\quad = \Delta^{\mathbf{D}}(\mathbf{C}\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \mathbf{C}\langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle) \\
&\quad = \Delta^{\mathbf{DC}}(\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle).
\end{aligned}$$

This concludes the proof. □

In order to directly link our results on pMAC to those of [AHM+15], the second relation we outline here, is the one between the distinguishing-type of game-based unforgeability definition for pMAC, UF-CMA from Definition 17, and the more "traditional" *game-winning* one, which we label gw-UF-CMA and informally describe next. Let $\mathbf{G}_{\mathsf{mac}}$ be a system that works as follows: on input a message $m$, output $\mathtt{Tag}_K(m)$, for a key $K$ initially sampled according to $\mathtt{Gen}$, and eventually accept one (final) forgery query $(m, \tau)$ as input. If an adversary $\mathbf{D}$ interacting with $\mathbf{G}_{\mathsf{mac}}$ submits a forgery $(m, \tau)$ which is both *new* (that is, $m$ was not queried before by $\mathbf{D}$) and *valid* (that is, $\mathtt{Vrf}_K(m, \tau) = m$), then we say that $\mathbf{D}$ wins the game $\mathbf{G}_{\mathsf{mac}}$. Finally, we define $\Gamma^{\mathbf{D}}(\mathbf{G}_{\mathsf{mac}})$ as the winning probability of $\mathbf{D}$, that is, the probability that $\mathbf{D}$ indeed submits a valid and new forgery to $\mathbf{G}_{\mathsf{mac}}$. Recall the classical result from the literature that, informally, asserts that distinguishing two systems is at most as hard as provoking an event in either one, such that the systems behave identically until this event is provoked. For the kind of systems we are considering, this was shown for example[16] in [Mau02, Theorem 1] and [MPR07, Lemma 4]. Then it is easy to see that indeed gw-UF-CMA-security implies UF-CMA-security, as we state next (without proof).

**Lemma 9.** *For every distinguisher* $\mathbf{D}$ *making at most* $q$ *verification queries,*

$$\Delta^{\mathbf{D}}(\langle \mathbf{T}_K, \mathbf{V}_K\rangle, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle) \leq q \cdot \Gamma^{\mathbf{DC}}(\mathbf{G}_{\mathsf{mac}}),$$

*for an appropriate reduction system* $\mathbf{C}$*. In particular, this implies that if a MAC scheme is* gw-UF-CMA*-secure, then it is also* UF-CMA*-secure.*

### C.3 Composable Anonymous Security of pMAC

After having introduced the game-based notion of key-indistinguishability for pMAC in [AHM+14], Alwen et al went on to define the corresponding composable

[16] For other kind of systems, such as *Code-Based Games*, the same result is usually referred to as the *"fundamental lemma of game-playing"* from [BR06, Lemma 2].

notion and relate this to the former in [AHM+15]. In this section we summarize on a high level how those results should be cast within our framework. For a fixed MAC scheme $\Sigma$, consider the $n$-protocol $\pi_{\mathsf{mac}} \doteq (\mathsf{tag}, \ldots, \mathsf{tag}, \mathsf{vrf})$, where, very informally, $\mathsf{tag}$ implements $\mathtt{Tag}_K$ and $\mathsf{vrf}$ executes $\mathtt{Vrf}_K$ on all keys and outputs the message and identity according to the index of the (unique) matching key (for more details see [AHM+15]). Then composable anonymous authenticity of a MAC scheme is defined as follows.

**Definition 20 (Composable Anonymous Security of pMAC).** *A MAC scheme $\Sigma$ achieves* composable anonymous authenticity *if*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}\times\mathcal{T}}] \xrightarrow{\pi_{\mathsf{mac}},\varepsilon} \mathsf{A\text{-}AUT}^n_{\mathcal{C}},$$

*that is, if there exists a simulator* $\mathsf{sim}$ *such that for all distinguishers* $\mathbf{D}$*,*

$$\Delta^{\mathbf{D}}(\pi_{\mathsf{enc}}[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}\times\mathcal{T}}], \mathsf{sim}^E\, \mathsf{A\text{-}AUT}^n_{\mathcal{C}}) \leq \varepsilon(\mathbf{D}).$$

Finally, we state the main theorem from [AHM+15] within our framework (without proof): a MAC scheme which is both key-indistinguishable and unforgeable implies the corresponding composable notion from Definition 20. We do so using our all-in-one anonymous security definition of UF-IK-CMA for pMAC instead of the two separate notions of unforgeability (UF-CMA) and key-indistinguishability (IK-CMA), as originally done by Alwen et al. By virtue of Lemma 8 and Lemma 9, our statement follows directly from their original proof.

**Theorem 8 ([AHM+15]).** *If a MAC scheme $\Sigma$ is* UF-IK-CMA*-secure, then it achieves composable anonymous authenticity, that is,*

$$[\mathsf{KEY}^n_{\mathcal{K}}, \mathsf{A\text{-}INS}^n_{\mathcal{C}\times\mathcal{T}}] \xrightarrow{\pi_{\mathsf{mac}},\varepsilon} \mathsf{A\text{-}AUT}^n_{\mathcal{C}},$$

*where* $\varepsilon(\mathbf{D}) \doteq \Delta^{\mathbf{DC}}([\langle \mathbf{T}_{K_1}, \mathbf{V}_{K_1}\rangle, \ldots, \langle \mathbf{T}_{K_n}, \mathbf{V}_{K_n}\rangle], \langle\langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle, \ldots, \langle \mathbf{T}_K, \mathbf{V}^{\perp}\rangle\rangle)$*, for an adequate reduction system* $\mathbf{C}$*.*