ETH Zürich, D-INFK
HS 2020, 4. December 2020

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Diskrete Mathematik

Exercise 13

Exercise 13.5 gives **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM20/`.

13.1 Free Variables ($\star$)

Determine all occurrences of free variables in the following formulas:

i) $\forall x \forall y \; (P(x,y) \lor P(x,z))$

ii) $\forall x \; (\exists x \; P(x) \land P(x)) \lor P(x)$

iii) $\forall x \; \big(\exists y \; P(y,x) \lor \exists z \; Q(x, f(z))\big)$

13.2 Interpretations ($\star$)

a) Which of the interpretations **i)**, **ii)** and **iii)** are models for the following formula? Justify your answers. (The symbol | denotes the divisibility relation.)

$$F = \forall x \forall y \forall z \; \big(P(f(x,y),x) \;\land\; P(f(x,y),y) \;\land\; \big(\neg P(x,y) \to \neg P(x, f(y,z))\big)\big)$$

i) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}$, $\quad f^{\mathcal{A}}(x,y) = x \cdot y$, $\quad P^{\mathcal{A}}(x,y) = 1 \iff y \mid x$

ii) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}$, $\quad f^{\mathcal{A}}(x,y) = x^y$, $\quad P^{\mathcal{A}}(x,y) = 1 \iff y \mid x$

iii) $U^{\mathcal{A}} = \mathcal{P}(\mathbb{N})$, $\quad f^{\mathcal{A}}(A,B) = A \cap B$, $\quad P^{\mathcal{A}}(A,B) = 1 \iff A \subseteq B$

b) Let $G = \big(\forall x \; \exists y \; P(x,y)\big) \;\land\; \big(\forall y \; \exists x \; P(x,y)\big) \;\land\; \big(\forall x \; \forall y \; (P(x,y) \to \neg P(y,x))\big)$. Find an interpretation with *a finite universe* that is

i) not suitable for G.

ii) suitable but not a model for G.

iii) suitable and a model for G.

13.3 Predicate Logic with Equality ($\star$)

We extend the syntax and the semantics of predicate logic as follows:

Syntax: If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula.

Semantics: If F is of the form $(t_1 = t_2)$ for terms t_1 and t_2, then $\mathcal{A}(F) = 1$ if and only if $\mathcal{A}(t_1) = \mathcal{A}(t_2)$.

a) Let $F = \forall x \forall y \; (x = y)$. Find the necessary and sufficient conditions for an interpretation $\mathcal{A}$ to be a model for F. Justify your answer.

b) Let $G = \exists x \exists y \; \neg(x = y)$. Find the necessary and sufficient conditions for an interpretation $\mathcal{A}$ to be a model for G. Justify your answer.

c) Find a formula with equality H, such that for any interpretation $\mathcal{A}$ suitable for H, we have $\mathcal{A}(H) = 1 \iff |U^{\mathcal{A}}| \geq 3$.

13.4 Statements About Formulas ($\star\star$)

Prove or disprove each of the following statements. Do not use any theorems or lemmas from the lecture notes. Note that x may appear free in F, G or both.

a) For any formulas F and G, we have

$$\forall x \; (F \wedge G) \models (\forall x \; F) \wedge G$$

b) For any formulas F and G, we have

$$\exists x \; (F \wedge G) \models (\exists x \; F) \wedge G$$

13.5 More Statements About Formulas ($\star\star$) *(8 Points)*

Which of the following statements are true for any formulas F and G? Prove your answer without using any lemmas or theorems from the lecture notes.

a) $\forall x \; (F \rightarrow G) \models (\forall x \; F) \rightarrow (\forall x \; G)$

b) $(\forall x \; F) \rightarrow (\forall x \; G) \models \forall x \; (F \rightarrow G)$

Due by 15. December 2020.
Exercise 13.5 is graded.