

# Cryptographic Protocols

Spring 2020

Part 3

## Proofs of Knowledge

Let  $Q(\cdot, \cdot)$  be a binary predicate and let a string  $z$  be given. Consider the problem of proving knowledge of a secret  $x$  such that  $Q(z, x) = \text{true}$ .

**Definition:** A protocol  $(P, V)$  is a **proof of knowledge for  $Q(\cdot, \cdot)$**  if the following holds:

- **Completeness:**  $V$  accepts when  $P$  has as input an  $x$  with  $Q(z, x) = \text{true}$ .
- **Soundness:** There exists an efficient program (knowledge extractor)  $K$ , which can interact with any program  $P'$  for which  $V$  accepts with noticeable (also called non-negligible) probability, and outputs a valid secret  $x$ .

**Note:**  $K$  can **rewind**  $P'$  (restart with same randomness).

## 2-Extractability

**Definition:** A three-move protocol (round) with challenge space  $C$  is **2-extractable** if from any two triples  $(t, c, r)$  and  $(t, c', r')$  with  $c \neq c'$  accepted by  $V$  one can efficiently compute an  $x$  with  $Q(z, x) = \text{true}$ .

**Theorem:** An interactive protocol consisting of  $s$  2-extractable rounds with uniform challenge from  $C$  is a proof of knowledge if  $1/|C|^s$  is negligible.

**Proof:** Knowledge extractor  $K$ :

1. Choose randomness for  $P'$  and execute the protocol between  $P'$  and  $V$ .
2. Execute the protocol again (same randomness for  $P'$ ).
- 3a. If  $V$  accepts in both executions, identify the first round with different challenges  $c$  and  $c'$  (but same  $t$ ). Use 2-extractability to compute an  $x$ , and output it (and stop).
- 3b. Otherwise, go back to Step 1.

## One-Way Group Homomorphisms (OWGH)

**Setting:** Groups  $\langle G, \star \rangle$  and  $\langle H, \otimes \rangle$

**Definition:** A **group homomorphism** is a function  $f$  with:

$$f : G \rightarrow H, \quad f(a \star b) = f(a) \otimes f(b)$$

**Notation:** We write  $[a]$  for  $f(a)$ , hence

$$[ ] : G \rightarrow H, \quad [a \star b] = [a] \otimes [b]$$

**Examples**

- $G = \langle \mathbb{Z}_q, + \rangle, H = \langle h \rangle$  with  $|H| = q, [a] = h^a$ :

$$[a + b] = h^{a+b} = h^a \cdot h^b = [a] \cdot [b]$$

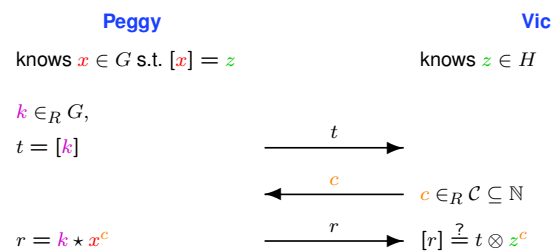
- $G = H = \langle \mathbb{Z}_m^*, \cdot \rangle, [a] = a^e$ :

$$[a \cdot b] = (a \cdot b)^e = a^e \cdot b^e = [a] \cdot [b].$$

## PoK of Pre-Image of OWGH – One Round of the Protocol

**Setting:** Groups  $G$  and  $H$ , group homomorphism  $[ ] : \langle G, \star \rangle \mapsto \langle H, \otimes \rangle$ .

**Goal:** Prove knowledge of a pre-image  $x$  of  $z \in H$ .



## 2-Extractability of OWGH PoK

**Theorem 3:** The protocol round is 2-extractable if

$$\exists \ell \in \mathbb{Z}, u \in G \text{ s.t. } (1) \forall c_1, c_2 \in C, c_1 \neq c_2 : \gcd(c_1 - c_2, \ell) = 1$$

$$(2) [u] = z^\ell$$

**Proof:** Given  $\ell$  and  $u$  as above and triples  $(t, c_1, r_1)$  and  $(t, c_2, r_2)$  with  $c_1 \neq c_2$  satisfying the verification test, extract  $x'$  with  $[x'] = z$  as follows:

$$1. \quad \begin{array}{l} [r_1] = t \otimes z^{c_1} \\ [r_2] = t \otimes z^{c_2} \\ \hline [r_2^{-1} \star r_1] = z^{c_1 - c_2} \end{array}$$

2. Extended Euclidean Algorithm  $\Rightarrow a, b$  with  $a\ell + b(c_1 - c_2) = 1$

$$3. \quad z = z^1 = z^{a\ell + b(c_1 - c_2)} = z^{a\ell} \otimes z^{b(c_1 - c_2)}$$

$$= (z^\ell)^a \otimes (z^{c_1 - c_2})^b = [u]^a \otimes [r_2^{-1} \star r_1]^b = \underbrace{[u^a \star (r_2^{-1} \star r_1)^b]}_{x'}$$

## OWGH PoK for Schnorr and Guillou-Quisquater

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### Schnorr

- $G = \mathbb{Z}_q$ , cyclic group  $H = \langle h \rangle$ ,  $|H| = q$  prime
- $[\ ] : G \rightarrow H$ ,  $x \mapsto [x] = h^x$ .
- Thm 1.5:  $\ell = q$ ,  $u = 0$ :  $z^\ell = 1 = [0]$ ;  $q$  prime  $\Rightarrow \gcd(c_1 - c_2, \ell) = 1$ .

### Guillou-Quisquater

- $G = H = \mathbb{Z}_m^*$ .
- $[\ ] : G \rightarrow H$ ,  $x \mapsto [x] = x^e$ .
- Thm 1.5:  $\ell = e$ ,  $u = z$ :  $z^\ell = z^e = [z]$ ;  $e$  prime  $\Rightarrow \gcd(c_1 - c_2, \ell) = 1$ .

### Further Examples

- see paper, lecture, and exercise.