Cryptographic Protocols

Spring 2020

MPC Part 3

CTP/CMP Homomorphic Commitments

Commitment Transfer

0. P_i committed to $a \Rightarrow P_j$ committed to a.
1. P_i sends opening information to P_j.

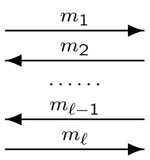
Commitment Multiplication

0. P_i com. to $A = [a, \alpha]$, $B = [b, \beta] \Rightarrow P_i$ com. to $C = [ab, \gamma]$.
1. P_i computes ab and commits to it.
2. P_i proves knowledge of a, α, ξ s.t. $A = [a, \alpha]$, $C = B^a \cdot [0, \xi]$.

Cryptographic Active MPC – Impossibility

Two Parties ($n = 2, t = 1$)

Alice		**Bob**
Input a		Input b
Output $c = ab$		Output $c = ab$

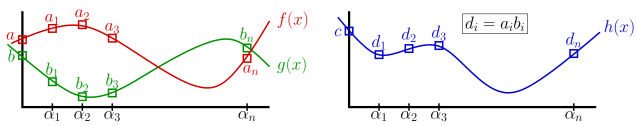


Analysis

- ℓ is the minimal length secure protocol.
- The protocol where m_ℓ is not sent is insecure (for Bob).
- Alice can cheat by withholding m_ℓ.

Generic Commitment Multiplication Proof

0. **Starting point**: D is committed to a, b, c by $\boxed{a}$, $\boxed{b}$, and $\boxed{c}$.
1. **CSP of a, b with degree t**
   $\Rightarrow f(x), g(x)$
2. **CSP of c with degree $2t$**
   use $h(x) = f(x)g(x)$



3. **Checks**
   $\forall P_i$: $d_i \stackrel{?}{=} a_i b_i$, broadcast accusation bit.
   On accusation: Open $\boxed{a_i}$, $\boxed{b_i}$, $\boxed{d_i}$, check $a_i b_i \stackrel{?}{=} d_i$.

Commit State

Open Protocol

1. **Open**
   D broadcasts $g(x)$.
2. **Check consistency**
   P_i accuses dealer if $g(\alpha_i) \neq s_i$.
3. **Compute secret**
   If $\leq t$ accusations: $s = g(0)$.
   If $> t$ accusations: disqualify dealer.

Proof:

Facts about two-dimensional polynomials (1/3)

Given: $f(x,y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x,y]$

Fact 1: $f_{y_0}(x) := f(x, y_0)$ is a one-dimensional polynomial of degree t.

Proof:
$$\begin{aligned} f(x,y_0) = & \ (f_{00} + f_{01}y_0 + \ldots + f_{0t}y_0^t) \\ & + (f_{10} + f_{11}y_0 + \ldots + f_{1t}y_0^t)x \\ & + \ldots \\ & + (f_{t0} + f_{t1}y_0 + \ldots + f_{tt}y_0^t)x^t \end{aligned}$$

Facts about two-dimensional polynomials (2/3)

Given: $f(x,y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x,y]$

Fact 2: Let $X = \{x_1, \ldots, x_{t+1}\}$ and $Y = \{y_1, \ldots, y_{t+1}\}$. Then $f(x,y)$ is uniquely defined by $W := \{(x_i, y_j, z_{ij}) \mid (x_i, y_j) \in X \times Y\}$.

Proof (existence): $\exists \geq 1$ such $f(x,y)$: Lagrange-Interpolation

Find $L_{ij}(x,y)$ with $\begin{cases} L_{ij}(x_i, y_j) = 1 \\ L_{ij}(x_{i'}, y_{j'}) = 0 \text{ for } (i',j') \neq (i,j) \end{cases}$

$$\Rightarrow \quad L_{ij}(x,y) := \prod_{\substack{i'=1 \\ i' \neq i}}^{t+1} \frac{x - x_{i'}}{x_i - x_{i'}} \prod_{\substack{j'=1 \\ j' \neq j}}^{t+1} \frac{y - y_{j'}}{y_j - y_{j'}}$$

and define
$$f(x,y) := \sum_{i,j=1}^{t+1} L_{ij}(x,y) z_{ij}.$$

Facts about two-dimensional polynomials (3/3)

Given: $f(x,y) = f_{00} + f_{10}x + f_{01}y + f_{11}xy + ... + f_{tt}x^t y^t \in \mathbb{F}[x,y]$

Fact 2: Let $X = \{x_1, \ldots, x_{t+1}\}$ and $Y = \{y_1, \ldots, y_{t+1}\}$. Then $f(x,y)$ is uniquely defined by $W := \{(x_i, y_j, z_{ij}) \mid (x_i, y_j) \in X \times Y\}$.

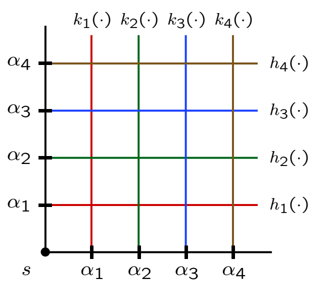
Proof (uniqueness): $\exists \leq 1$ such $f(x,y)$

1. Let $f_1(x,y)$ and $f_2(x,y)$ degree-t-polynomials through W.
2. $f_\triangle(x,y) := f_1(x,y) - f_2(x,y)$ is a degree-t-polynomial.
3. $\forall (x_i, y_j) \in X \times Y : f_\triangle(x_i, y_j) = 0$.
4. $\forall y_j \in Y : f_{y_j}(x) := f_\triangle(x, y_j)$ is a polynomial of degree t (Fact 1).
5. $\forall y_j \in Y : f_{y_j}(x_1) = f_{y_j}(x_2) = \ldots = f_{y_j}(x_{t+1}) = 0$; thus $f_{y_j} \equiv 0$.
6. Thus: $\forall (x, y_j) \in \mathbb{F} \times Y : f_\triangle(x, y_j) = 0$.
7. $\forall x \in \mathbb{F} : f_x(y) := f_\triangle(x,y)$ is a polynomial of degree t (Fact 1).
8. $\forall x \in \mathbb{F} : f_x(y_1) = f_x(y_2) = \ldots = f_x(y_{t+1}) = 0$; thus $f_x \equiv 0$.
9. Thus: $\forall (x,y) \in \mathbb{F} \times \mathbb{F} : f_\triangle(x,y) = 0$; thus $f_\triangle \equiv 0$.

Commit Protocol

1. **Distribution**
   D selects random polynomial
   $$f(x,y) = \sum_{i=0}^{t} \sum_{j=0}^{t} f_{ij}\, x^i y^j, \text{ with } f_{0,0} = s,$$
   and sends $h_i(x) = f(x, \alpha_i),\ k_i(y) = f(\alpha_i, y)$ to P_i.
2. **Consistency checks**
   $\forall P_i, P_j$: P_i sends $k_i(\alpha_j)$ to P_j, P_j complains if $k_i(\alpha_j) \neq h_j(\alpha_i)$.
   D broadcasts $f(\alpha_i, \alpha_j)$.
3. **Accusation**
   $\forall P_i$: if P_i has received contradicting values from D: accuse D.
   D broadcasts $h_i(x)$ and $k_i(y)$.
   Repeat until no further accusation.
4. **Compute share**
   If $> t$ accusing players: disqualify dealer.
   If $\leq t$ accusing players: $s_i = k_i(0)$.

Commit Protocol



$f(x,y)$,
$\deg(f) \leq t$

MPC Operations: Overview

	Passive	Active (commitment scheme)	Crypt. (normal commitments)	Inf.-Theo. (distributed commitments)
Input	Shamir sharing	+commit/CTP		
Add	linearity	+homomorph		
Mult.	mult.prot. $[n/2]$	+CMP/CTP	$B^a \sim C$	generic $[n/3]$
Output	interpolate	+CTP		$[n/3]$