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Diskrete Mathematik

Solution 2

2.1 A Proof

The way, in which the claim is stated is somewhat misleading. If one writes it down formally, using propositions, it becomes clear, that the claim consists in fact of two parts: "There exists the largest natural number n" and "$n = 1$". Let us denote the first of the above propositions by A and the second by B. Now the claim states that $A \land B$ is true. However, in the proof we already assumed that n exists. Therefore, we only showed that $A \to B$ is a tautology. In other words, we proved that "If the largest natural number exists, then it is equal to 1."

As a side note, it is also possible to prove, in a similar way, more statements of this sort, for example "If the largest natural number exists, then it is equal to 0.".

2.2 Propositional Logic

a) The formulas can be stated in the English language in the following way:

- **i)** F_1: "The monkey has a banana and does not sit on the palm tree."
- **ii)** F_2: "The monkey sits on the palm tree and has a banana, or it does not sit on the palm tree and it does not have a banana."
  Equivalently, we could say "The monkey sits on the palm tree if and only if it has a banana."

b) The sentences can be written formally in the following way:

- **i)** $F_3 := \neg A \land \neg B$
- **ii)** $F_4 := (\neg A \land B) \lor (A \land \neg B)$

c)

- **i)** $\neg F_3$: The monkey sits on the palm tree or it has a banana.

$$\neg F_3 \equiv \neg(\neg A \land \neg B) \equiv A \lor B$$

- **ii)** $\neg F_4$: The monkey sits on the palm tree if and only if it has a banana.

$$\neg F_4 \equiv (A \land B) \lor (\neg A \land \neg B) \equiv F_2$$

2.3 Function Tables and Equivalence

a)

A	B	C	$B \to C$	$\neg(A \to C) \wedge \neg(A \vee B)$	$(B \to C) \to \big(\neg(A \to C) \wedge \neg(A \vee B)\big)$
0	0	0	1	0	0
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	1	0	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	0	0	1
1	1	1	1	0	0

b) With the above function table, it becomes clear that the formula in a) is true if and only if $B \wedge \neg C$ is true. Therefore, the simple equivalent formula is $B \wedge \neg C$.

c) We have:

$$
\begin{aligned}
& (B \to C) \to \big(\neg(A \to C) \wedge \neg(A \vee B)\big) & \\
\equiv\ & (B \to C) \to \big(\neg(\neg A \vee C) \wedge \neg(A \vee B)\big) & \text{(definition of } \to \text{)} \\
\equiv\ & (B \to C) \to \big(\big(\neg(\neg A) \wedge \neg C\big) \wedge \neg(A \vee B)\big) & (\ \neg(F \vee G) \equiv \neg F \wedge \neg G\ ) \\
\equiv\ & (B \to C) \to \big(\big(\neg(\neg A) \wedge \neg C\big) \wedge (\neg A \wedge \neg B)\big) & (\ \neg(F \vee G) \equiv \neg F \wedge \neg G\ ) \\
\equiv\ & (B \to C) \to \big((A \wedge \neg C) \wedge (\neg A \wedge \neg B)\big) & \text{(double negation)} \\
\equiv\ & (B \to C) \to \big(A \wedge \neg C \wedge \neg A \wedge \neg B\big) & \text{(associativity of } \wedge\text{)} \\
\equiv\ & (B \to C) \to \big(\neg C \wedge \neg B \wedge A \wedge \neg A\big) & \text{(commutativity of } \wedge\text{)} \\
\equiv\ & (B \to C) \to \big((\neg C \wedge \neg B) \wedge (A \wedge \neg A)\big) & \text{(associativity of } \wedge\text{)} \\
\equiv\ & (B \to C) \to \big((\neg C \wedge \neg B) \wedge \bot\big) & (\ F \wedge \neg F \equiv \bot\ ) \\
\equiv\ & (B \to C) \to \bot & (\ F \wedge \bot \equiv \bot\ ) \\
\equiv\ & \neg(B \to C) \vee \bot & \text{(definition of } \to \text{)} \\
\equiv\ & \neg(B \to C) & (\ F \vee \bot \equiv F\ ) \\
\equiv\ & \neg(\neg B \vee C) & \text{(definition of } \to \text{)} \\
\equiv\ & \neg(\neg B) \wedge \neg C & (\ \neg(F \wedge G) \equiv \neg F \vee \neg G\ ) \\
\equiv\ & B \wedge \neg C & \text{(double negation)}
\end{aligned}
$$

2.4 Logical Consequence

a) We first construct the function table for the formula $A \wedge (A \to B)$. *(1 Point)*

A	B	$A \wedge (A \to B)$
0	0	0
0	1	0
1	0	0
1	1	1

The above table shows that the truth value of $A \wedge (A \to B)$ is 1 only for the truth assignment in the last row. Clearly, B is also true for that assignment. Thus, B is the logical consequence of $A \wedge (A \to B)$ and the statement holds. *(1 Point)*

b) The statement is false. There exists a truth assignment, namely one in which A is false and B is true, for which $A \to B$ is true, but $\neg A \to \neg B$ is false. *(1 Point)*

Thus, $\neg A \to \neg B$ is not a logical consequence of $A \to B$. *(1 Point)*

c) We first construct the function table for both formulas: $(A \to B) \wedge (B \to C)$ and $A \to C$. *(2 Points)*

A	B	C	$A \to B$	$B \to C$	$(A \to B) \wedge (B \to C)$	$A \to C$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Analogously to the subtask a), we can show that the statement holds. *(1 Point)*

2.5 Satisfiability and Tautologies

a) This formula is satisfiable, since it is true for the assignment $A = 0, B = 1$. It is, however, not a tautology, since it is false for the assignment $A = 0, B = 0$.

b) This formula is not satisfiable (hence, it is also not a tautology). In order to justify this claim, let $F := \big((A \to B) \wedge (B \to C)\big) \wedge \neg(A \to C)$. By Lemma 2.2, F is unsatisfiable if and only if $\neg F$ is a tautology. We have $\neg F \equiv (A \to B) \wedge (B \to C) \ \to \ (A \to C)$. From the exercise 2.4 c), we know that $(A \to B) \wedge (B \to C) \ \models \ (A \to C)$ holds. Form this fact, together with Lemma 2.3, it follows that $\neg F$ is a tautology. Thus, F is unsatisfiable.

2.6 Knights and Knaves

Let A be the proposition "The islander is a knight." and let B be the proposition "The left road leads to the village.". We want to determine, whether B is true. To this end, we would like to find a formula that is true exactly when (1) both A and B are true (in such case, the man is indeed a knight and he answers truthfully about B) and (2) both A and B are false (this time, the man is a knave, lies about B being false, and thus tells the truth about B). One possible such formula would be $(A \wedge B) \vee (\neg A \wedge \neg B)$. Formulated as a question, this would mean "Are you a knight and the left road leads to the village, or is it the case, that you are a knave and the left road leads to the jungle?".

Another possible question would be "Would the member of the group, to which you don't belong say that the right road leads to the village?". Also here, the left road leads to the village, if the islander answers "Yes.".